58. On Locally Q-complete Spaces. II

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1. In this paper we shall consider the problems characterizing a given space X by a ring of continuous functions defined on X.[1] Shirota [1] has proved that if X is a Q-space, then $C(X)$ characterizes X, that is, the ring isomorphism of $C(X)$ onto $C(Y)$ implies that X is homeomorphic to Y for any Q-spaces X and Y. In general, it is easy to see that $C(X)$ or $B(X)$ does not characterize X. But under some conditions on a ring isomorphism this problem is solved in the affirmative [2, 6]. On the other hand, Shanks [3] and Ishii [4] have proved that if X is locally compact, then $C_k(X)$[2] characterizes X.

In this paper, we shall generalize Shanks' theorem and it will be shown that for any locally Q-complete space X which is not compact, there is a subring of $C(X)$ on which any non-trivial ring homomorphism[3] is a point ring homomorphism. Moreover we shall prove that such a subring characterizes X.

2. Extension of functions

Let $f \in C(X)$ and $\tilde{f}$ be a continuous extension over βX of f (if it exists, i.e. f is bounded). If f can be continuously extended over a point $p \in \beta X - X$, f has a finite value at the point p. If f is not continuously extended over the point p, then for any $m>0$, $f_m = (f \wedge m) \vee (-m)$[4] has a continuous extension $\tilde{f}_m$ because f_m is bounded. It is easily seen that $\tilde{f}_m(p) = m$. Therefore there exists a neighborhood (in X)[5] of the point p on which $f > n$ for a given integer $n > 0$. Let

1) A space X considered here is always a completely regular T_1-space, and other terminologies used here, for instance $C(X)$, are the same as in [7].

2) $C_k(X)$ is a ring consisting of all continuous functions which have compact supports.

3) A non-trivial *ring homomorphism* of a subring C_1 of $C(X)$ means a ring homomorphism of C_1 *onto* R where R is a ring of all real numbers. But a ring homomorphism is not necessarily *linear*, for C_1 does not necessarily contain constant functions. A point ring homomorphism φ is defined by $\varphi(f) = f(p)$ for all $f \in C_1$ where p is a fixed point in X. In this case φ is completely determined by the point p, and hence we shall write $\varphi = \varphi_p$. A ring homomorphism φ is called to be *trivial* if $\varphi(f) = 0$ for all $f \in C_1$.

4) For any constant m, where no confusion will arise, we use the same letter m for a function which takes a constant value m on X. The symbols "$\vee$" and "$\wedge$" are used in the following sense:

$$(f \vee g)(x) = \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) = \min(f(x), g(x)).$$

5) *A neighborhood* (*in* X) *of* x^* means a set U such that $U = X \frown V$ where V is a neighborhood of x^* in βX.

$C(f)=X\smile\{x;\ x\in\beta X-X,\ f$ can be continuously extended over $\{x\}\}$.

From these arguments we have

Lemma 1. *Let $f\in C(X)$ and $x^*\notin C(f)$; then there is a neighborhood of x^* (in X) on which $f>n$ for any given integer $n>0$.*

Lemma 2. *Under the same conditions as in Lemma 1, if we put $g=f/\max(q, f^2)$ for any $q>1$, then we have $\tilde{g}(x^*)=0$.*

Proof. Since g is bounded, g has a continuous extension $\tilde{g}$. $C(f)\not\ni x^*$ implies that for any $m>q$, there is a neighborhood U(in X) on which $f>m$ by Lemma 1. Therefore $g|U=1/f<1/m$. This means that $\tilde{g}(x^*)=0$.

Lemma 3. *Under the same conditions as in Lemma 1, we have $\widetilde{(fg)}(x^*)=1$ for any $q>1$.*

Proof. For any $q>1$, the set $A=\{x;\ |f(x)|\geq q\}$ is not void because f is not bounded. By the definition of g, it is obvious that $fg=1$ on A. Since $x^*\notin C(f)$, A contains some neighborhood (in X) of x^* on which $f>q$ by Lemma 1. Therefore it is easily verified that $\widetilde{(fg)}(x^*)=1$.

3. Subring $C_B(X)$

In §§ 3 and 4, we assume that X is locally Q-complete but not compact and B is a compact subset of βX contained in $\beta X-X$ such that i) in case X is a Q-space, B is any compact subset, ii) in case X is not a Q-space, B is any compact subset containing $(\nu X-X)^\beta$.[6] In case ii) B is disjoint from X because X is open in νX by Theorem 2 in [7]. Let us put $Y=\beta X-B$, and $C_B(X)$ be a set of all functions in $C(X)$ which have the property such that $Z(f)^\beta$ contains a neighborhood (in βX) of B where $Z(f)=\{x;\ f(x)=0,\ x\in X\}$.

Lemma 4. *If $f, g\in C_B(X)$, then $f+g\in C_B(X)$.*

Proof. Suppose that $Z(f)^\beta$ (or $Z(g)^\beta$) contains an open neighborhood U(in βX) (or V(in βX)) of B. $U\frown V$ is an open neighborhood (in βX) of B and $W=X\frown(U\frown V)$ is a non-void open subset of X, because X is dense in βX. By the definition, both f and g vanish on W. Since $(U\frown V)$ is open and X is dense in βX, it is obvious that $W^\beta\supset U\frown V$, and hence $Z(f+g)^\beta\supset W^\beta\supset U\frown V$. This means that $f+g\in C_B(X)$.

If $C_B(X)\ni f$, then for any $g\in C(X)$, it is easily verified that $fg\in C_B(X)$. Thus $C_B(X)$ is an ideal of $C(X)$. On the other hand, $C_k(Y)$ is considered as a subring contained in $C_B(X)$, since X is dense in Y and Y is locally compact.

Theorem 1. *Let X be locally Q-complete but not compact. If B is a compact subset of βX contained in $\beta X-X$ such that* i) *in case X is a Q-space B is any compact subset,* ii) *in case X is not a Q-space B is any compact subset containing $(\nu X-X)$, then any non-trivial*

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6) A^β denotes a closure (in βX) of A where A is any subset.

ring homomorphism of $C_B(X)$ is a point ring homomorphism.

Proof. If X is pseudo-compact, then we have $\nu X - X = \beta X - X$, that is, $C_B(X) = C_k(X)$, and hence we can assume that X is not pseudo-compact. Let φ be a non-trivial ring homomorphism of $C = C_B(X)$; then φ can be regarded as a non-trivial ring homomorphism of $C_k(Y)$ where $Y = \beta X - B$. For if $\varphi = 0$ on $C_k(Y)$, then for any $f \in C$ there exists a $g \in C_k(Y)$ such that $g = 1$ on $\{x; f(x) \neq 0\}$. Therefore we have $\varphi(f) = \varphi(fg) = \varphi(f)\varphi(g) = \varphi(f) \cdot 0 = 0$. This means that $\varphi = 0$ on C. Therefore, by Theorem 5 (Ishii [4]) φ becomes a point ring homomorphism of $C_k(Y)$, that is, there is a point x^* in Y such that $\varphi_{x^*} = \varphi$, i.e. $\varphi(f) = f(x^*)$ for all $f \in C_k(Y)$. Let f be any function in C. Since $g = f/\max(1, f^2)$ is bounded and $g = 0$ on $Z(f)$, we can consider g as a function of $C_k(Y)$. Similarly fg is also regarded as a function of $C_k(Y)$. From the remark above and the fact that φ is a ring homomorphism, we have $\varphi(fg) = \widetilde{(fg)}(x^*)$ and $\varphi(fg) = \varphi(f)\varphi(g) = \varphi(f)\tilde{g}(x^*)$. Now suppose that $x^* \in Y - X$ and $C(f) \not\ni x^*$. By Lemmas 2 and 3 we have $\widetilde{(fg)}(x^*) = 1$ and $\tilde{g}(x^*) = 0$. This is a contradiction. Thus either X contains x^* or $C(f) \ni x^*$. In case X contains x^*, then φ is a point ring homomorphism, because for any $g \in C - C_k(Y)$, we take a function k in $C_k(Y)$ such that $k(x^*) = 1$ and $k(x) = 0$ for $x \notin \{y; g(x^*) - 1 < g(y) < g(x^*) + 1\} \frown U \frown X$ where U is a neighborhood of x^* which is disjoint from a neighborhood (in βX) of B. Then $kg \in C_B(X)$ and $\varphi(kg) = \widetilde{(kg)}(x^*) = g(x^*)$. On the other hand, $\varphi(kg) = \varphi(k)\varphi(g) = k(x^*)\varphi(g)$. This means that $g(x^*) = \varphi(g)$. Therefore we shall consider the remainder case: $Y - X$ contains x^* and $C(f) \ni x^*$ for all $f \in C$. But in the following we shall prove that this case does not happen. Since $Y - X \subset \beta X - B - X \subset \nu X$, we have $(Y - X) \frown (\nu X - X) = \theta$, that is, $\nu X - X \not\ni x^*$. By (v) in [5], x^* is contained in G_δ-set of βX which is disjoint from νX. Therefore there is a function $f \in B(X)$ such that $\tilde{f}(x^*) = 0$ and $f > 0$ on X. On the other hand, βX is normal, there is a function h on βX such that $h(B) = -1$ and $h(x^*) = 1$. It is easy to see that $((h|X) \vee 0)/f$ is not bounded on X. By the method of construction of h, $((h|X) \vee 0)/f$ is a function contained in $C_B(X)$, and hence we have proved that there is a non-bounded function in $C_B(X)$ which can not be continuously extended over the point x^*. Thus the ring homomorphism φ must be a point ring homomorphism φ_{x^*}, $x^* \in X$.

Theorem 1 shows that there are no maximal ideals, except fixed maximal ideals, by which the residue class rings are isomorphic to the ring of all real numbers.

Next we shall introduce a topology in $\hat{X}$ which is a set of all fixed maximal ideals in $C_B(X)$ as follows:

$$Cl(\hat{A}) \ni I(a) \leftrightarrow \bigcap_{x \in A} I(x) \quad I(a)$$

where $\widehat{A}=\{I(x); x\in A\subset X\}$, and $I(x)$ denotes a maximal ideal whose element vanishes at the point x. Then it is easily seen, using the same method as in [6], that the mapping $x\to I(x)$ gives a homeomorphism of X onto $\widehat{X}$ [6].

From Theorem 1 and the definition of topology of $\widehat{X}$, we have

Theorem 2. *Let X be locally Q-complete but not compact and let B be any compact subset contained in $\beta X-X$ such that* i) *in case X is a Q-space, B is any compact subset,* ii) *in case X is not a Q-space, B is any compact subset containing $(\nu X-X)^{\beta}$; then $C_B(X)$ determines X.*

Any Q-space is locally compact, and moreover any locally compact space is always locally Q-complete [7]. Thus we have obtained a subring of $C(X)$ which determines X, for any locally Q-complete space which is not compact.

4. Subring $C_\nu(X)$

In this section we shall moreover assume that X is not a Q-space. We denote by $C_\nu(X)$ the subring of $C(X)$ whose extension over νX vanishes on $\nu X-X$. Then we have

i) *$Z(f)\neq\theta$ for any $f\in C_\nu(X)$.* For if $Z(f)=\theta$, then $1/f\in C(X)$ but $1/f$ has not a continuous extension over νX because $f(\nu X-X)=0$. This is a contradiction.

ii) *$Z(f)^{\beta}$ contains $B=(\nu X-X)^{\beta}$ for any $f\in C_\nu(X)$.* We assume that, no loss of generality, that $f\geq 0$. It is sufficient to prove that $Z(f)^{\beta}\supset\nu X-X$. If there is a point $b\in\nu X-X-Z(f)^{\beta}$, there is a function g in $C(\nu X)$ such that $g(b)=0$, $g(Z(f)^{\beta}\cup\nu X)=1$, and f is positive on some neighborhood of b. Then $f+g$ is positive on X and $f+g$ has an extension over νX and $(f+g)(b)=0$. On the other hand, $1/(f+g)\in C(X)$ and it has a continuous extension over νX. This is a contradiction.

iii) *If $C_B(X)=C_\nu(X)$, then $\nu X-X$ is compact.* Suppose that there is a point b in $B-(\nu X-X)$. Since νX is a Q-space, there is a continuous function on βX such that $f(b)=0$ and f is positive on X because b is contained in a G_δ-set of βX which is disjoint from νX [5]. On the other hand, since βX is compact, there is a continuous function g such that $g(B)=0$ and g is not identically zero on X. Then $f+g\in C(X)$ but $Z(f+g)^{\beta}$ contains no neighborhoods of B by the method of construction of f. Therefore $\nu X-X$ coincides with B and hence it is compact.

The converse of iii) does not hold. Such an example is given by the following space X.

Example. $X=[1,\Omega]\times[1,\omega]-(\Omega,\omega)$ where ω and Ω are the first ordinals of the second and the third classes respectively. Then X is pseudo-compact and locally compact moreover $\beta X=X\cup\{(\Omega,\omega)\}$. Thus

$C_B(X)=C_k(X)$ and $C_\nu(X)$ contains a continuous function g defined by $g(\alpha, n)=1/n$ and $g(\alpha, \omega)=0$ where $1\leq\alpha\leq\Omega$. It is obvious that $C_B(X)$ does not contain g, and hence $C_B(X)\neq C_\nu(X)$ even if $\nu X-X$ consists of only one point (Ω, ω) (and hence compact).

We notice that the point $p=(\Omega, \omega)$ is not a P-point[7] of βX and in this case, βX is considered as a natural one-point Q-completion of X.

Let X_ν be the natural one-point Q-completion of X and p be an adjointed point, i.e. $X_\nu=X\cup\{p\}$ (see [7]).

Theorem 3. *Suppose that X is locally Q-complete but not a Q-space. Then $C_\nu(X)=C_B(X)$ if and only if an adjointed point p of the natural one-point Q-completion X_ν of X is a P-point of X_ν where $B=(\nu X-X)^\beta$.*

Proof. $C_\nu(X)$ can be regarded as a subset of $C(X_\nu)$ consisting of all elements of $C(X_\nu)$ which vanish at the point p. Therefore it is easily verified that if $C_\nu(X)=C_B(X)$, then p is a P-point of X_ν. Conversely, if p is a P-point of X_ν, then for each $f\in C_\nu(X)$, $\overline{Z(f)}$(in $X\cup B$) contains a neighborhood of B in $X\cup B$. Since $Z(f_m)=Z(f)$ for some $m>0$, it is easy to see that $Z(f)^\beta$ contains a neighborhood (in βX) of B, and hence we have that $C_\nu(X)=C_B(X)$.

From Theorem 3 and ii) we have

Corollary. *If p is a P-point of X_ν, $\nu X-X$ is compact.*

References

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7) A point p of X is said to be a P-point of X if every continuous function which vanishes at p vanishes on a neighborhood of p.