84. *On the Sets of Regular Measures. II*

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Theorem 5. (1) Let $\nu=\bigcap_{\lambda\in\Lambda}\mu_\lambda$ be the inferior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Then, if any measurable set E is inner regular with respect to each μ_λ, $\lambda\in\Lambda$ satisfying $\mu_\lambda(E)<\infty$, the measurable set of ν-finite measure is inner regular with respect to ν, too.

(2) Let μ and ν be two measures. Then, if μ is σ-finite and outer (inner) regular, $\nu\leqq\mu$ implies the strictly outer (inner but not necessarily strictly inner) regularity of ν. (These results will be applied, for instance, to the case when ν is the inferior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$ and at least one measure μ_λ, $\lambda\in\Lambda$ is σ-finite and outer (inner) regular.)

Proof. (1) If $\nu(E)<\infty$, there exist (refer to (1) of Theorem 4) a sequence, $\{\lambda_i\}_{i=1}^\infty$, and a partition $\{A_i\}_{i=1}^\infty$ of E such that $\lambda_i\in\Lambda$ $(i=1,2,\cdots)$, $\bigcup_{i=1}^\infty A_i=E$, $A_j\frown A_k=\theta$ $(j\neq k)$, $A_i\in S$ $(i=1,2,\cdots)$ and $\nu(E)\leqq\mu_{\lambda_1}(A_1)+\mu_{\lambda_2}(A_2)+\cdots+\mu_{\lambda_i}(A_i)+\cdots<\infty$. For an arbitrary $\varepsilon>0$, let C_i be a compact measurable set contained in A_i such that $\mu_{\lambda_i}(C_i)>\mu_{\lambda_i}(A_i)-\varepsilon/2^{i+1}$ $(i=1,2,\cdots)$ and let $C=\bigcup_{i=1}^\infty C_i$. Then $C\subseteqq E$ and $\nu(E-C)\leqq\nu(A_1-C_1)+\nu(A_2-C_2)+\cdots+\nu(A_i-C_i)+\cdots\leqq\mu_{\lambda_1}(A_1-C_1)+\mu_{\lambda_2}(A_2-C_2)+\cdots+\mu_{\lambda_i}(A_i-C_i)+\cdots<\varepsilon/2$. Therefore $\nu(\bigcup_{i=1}^N C_i)>\nu(E)-\varepsilon$ for a suitable integer N.

(2) The assumptions of the σ-finiteness and the outer regularity of μ imply clearly the strictly outer regularity of μ, therefore any measure ν such as $\nu\leqq\mu$ is also naturally strictly outer regular.

Next, suppose that μ is σ-finite and inner regular. In this case, there exists a σ-compact, measurable set $C=\bigcup_{i=1}^\infty C_i$ such that $C\subseteqq E$ and $\mu(E-C)<\varepsilon/2$, $\nu(E-C)<\varepsilon/2$ for an arbitrary measurable set E and an arbitrary $\varepsilon>0$.

Now we distinguish two cases:

I. $\nu(E)<\infty$. In this instance, $\nu(C-\bigcup_{i=1}^N C_i)<\varepsilon/2$, hence $\nu(E-\bigcup_{i=1}^N C_i)<\varepsilon$ for a suitable integer N.

II. $\nu(E)=\infty$. It follows $\nu(C)=\infty$ and there exists an integer N such that $\nu(\bigcup_{i=1}^N C_i)>M$ for an arbitrary $M>0$.

Remark 1. The following examples show that situations with respect to outer and inner reguralities are not parallel.

Example 1. This shows the falsity of the more general statement than (2) of Theorem 4: if μ_1 and μ_2 are inner regular, then $\nu=\mu_1\bigcap\mu_2$ is also inner regular.

Let X_1 and X_2 be two non-countable sets such that $X_1\frown X_2=\theta$

and let $X=X_1\smile X_2$ be a discrete space. Denote the classes of all (at most) countable sets of X and of all complementary (in X) sets of the countable sets by the symbols S_1 and S_2, respectively, and let $S=S_1\smile S_2$. Surely, $X_1\notin S_1$, $X_2\notin S_2$, and S will be a σ-algebra. For every point a in X, let $\mu_1(\{a\})$ and $\mu_2(\{a\})$ be as follows:

$$\mu_1(\{a\})=1,\ \mu_2(\{a\})=0\ (a\in X_1);\ \mu_1(\{a\})=0,\ \mu_2(\{a\})=1\ (a\in X_2),$$

and for every $E\in S_2$ let $\mu_1(E)=\mu_2(E)=\infty$. Then, $\mu_1(E)$ and $\mu_2(E)$ will be the numbers of points in $E\frown X_1$ and $E\frown X_2$ respectively, and moreover, $\nu(E)=0$ or $\nu(E)=\infty$ according as $E\in S_1$ or $E\in S_2$. Certainly, although μ_1 and μ_2 are inner regular, every $E\in S_2$ fails to be inner regular with respect to ν.

Consequently, in the above case, it is impossible to express μ_1 and μ_2 as the integral measures with respect to a common measure (refer to (2) of Theorem 4).

Example 2. This shows the falsity of the statement similar to (1) of Theorem 3: if a certain set $E\in S$ is outer regular with respect to μ_1 and μ_2, E is outer regular with respect to $\nu=\mu_1\frown\mu_2$, too.

The following counter-example may be formed even on condition that $\mu_1=\int f_1 d\mu$, $\mu_2=\int f_2 d\mu$.

Let $X_1=\{a_i\}_{i=1}^{\infty}$ be a countable set and X_2 be a non-countable set disjoint to X_1, and let $X=X_1\smile X_2$ be a concrete space. The σ-algebra generated by $\{a_1\}, \{a_2\},\cdots, \{a_i\},\cdots$ and X_2 will be denoted by S. We shall introduce the measures μ_1 and μ_2 by the following identities:

$$\left.\begin{array}{l}\mu_1(\{a_i\})=\frac{1}{i},\ \mu_2(\{a_i\})=\frac{1}{i^2}\ (i\text{:odd})\\ \mu_1(\{a_i\})=\frac{1}{i^2},\ \mu_2(\{a_i\})=\frac{1}{i}\ (i\text{:even})\end{array}\right\};\ \mu_1(X_2)=\mu_2(X_2)=c,\ 0<c\leqq\infty.$$

Then, $\mu_1(X_1)=\mu_2(X_1)=\infty$, hence X_1 is outer regular with respect to μ_1 and μ_2. On the other hand, $\nu(X_1)=(\mu_1\frown\mu_2)(X_1)=\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty$ and $\nu(X)=\nu(X_1)+\nu(X_2)=\sum_{i=1}^{\infty}\frac{1}{i^2}+c>\sum_{i=1}^{\infty}\frac{1}{i^2}=\nu(X_1)$. Therefore, X_1 is not outer regular with respect to ν, because X is the only one open measurable set containing X_1.

By the way, if we define a measure μ and two measurable functions $f_1(x)$, $f_2(x)$ in such ways that $\mu(\{a_i\})=1$ $(i=1, 2,\cdots)$, $\mu(X_2)=1$ and $f_1(a_i)=\frac{1}{i}$ $(i\text{: odd})$, $f_1(a_i)=\frac{1}{i^2}$ $(i\text{: even})$, $f_1(x)=c$ $(x\in X_2)$, $f_2(a_i)=\frac{1}{i^2}$ $(i\text{: odd})$, $f_2(a_i)=\frac{1}{i}$ $(i\text{: even})$, $f_2(x)=c$ $(x\in X_2)$, then $\mu_1=\int f_1 d\mu$ and $\mu_2=\int f_2 d\mu$.

4. Integral measures. Let μ_1, μ_2 and ν be three measures of the following types: $\mu_1=\int f_1 d\mu$, $\mu_2=\int f_2 d\mu$, $\nu=\int\sqrt{f_1 f_2}\,d\mu$, where f_1 and f_2

are both non-negative measurable functions and μ is a certain measure.

Now, the outer (inner) regularities of μ_1 and μ_2 do not necessarily imply that of ν, and in this connection, a counter-example will be set forth afterward.

Well, we shall propose a certain sufficient condition.

Theorem 6. Let $\mu_1(E)$ and $\mu_2(E)$ be finite or infinite simultaneously for every $E \in S$ (Property (A)).

(1) If a set $E \in S$ is outer regular with respect to μ_1 and μ_2, then E is outer regular with respect to ν, too.

(2) If μ_1 and μ_2 are inner regular, then ν is also inner regular.

Proof. Denote the sets $\{x : f_1(x) \leqq f_2(x)\}$ and $\{x : f_1(x) > f_2(x)\}$ by X_1 and X_2, respectively.

(1) The following three inequalities will be of use for the arguments:

$$\nu(F) \leqq \frac{1}{2}\{\mu_1(F)+\mu_2(F)\} \ (F \in S), \quad \left.\begin{matrix} \mu_1(E\frown X_1) \leqq \nu(E\frown X_1) \leqq \mu_2(E\frown X_1) \\ \mu_2(E\frown X_2) \leqq \nu(E\frown X_2) \leqq \mu_1(E\frown X_2) \end{matrix}\right\}.$$

We need consider only the case $\nu(E) < \infty$. In this instance, $\mu_1(E\frown X_1) < \infty$, $\mu_2(E\frown X_2) < \infty$ and Property (A) imply $\mu_2(E\frown X_1) < \infty$, $\mu_1(E\frown X_2) < \infty$, and accordingly, $\mu_1(E) < \infty$, $\mu_2(E) < \infty$ hold. Therefore, there exist the two open measurable sets U_1 and U_2 such that $U_1 \supseteqq E$, $U_2 \supseteqq E$ and $\mu_1(U_1 - E) < \varepsilon$, $\mu_2(U_2 - E) < \varepsilon$ for an arbitrary $\varepsilon > 0$; thus, $\nu((U_1 \frown U_2) - E) \leqq \frac{1}{2}\{\mu_1((U_1 \frown U_2) - E) + \mu_2((U_1 \frown U_2) - E)\} < \varepsilon$ and the outer regularity of E with respect to ν results.

(2) Let E be any set belonging to S. It was already indicated that $\mu_1(E) < \infty$ and $\mu_2(E) < \infty$ in case $\nu(E) < \infty$. Therefore, there exist the two compact measurable sets C_1 and C_2 such that $C_1 \subseteqq E$; $C_2 \subseteqq E$ and $\mu_1(E - C_1) < \varepsilon$, $\mu_2(E - C_2) < \varepsilon$ for an arbitrary $\varepsilon > 0$. This implies $\nu(E - (C_1 \smile C_2)) \leqq \frac{1}{2}\{\mu_1(E - (C_1 \smile C_2)) + \mu_2(E - (C_1 \smile C_2))\} < \varepsilon$, and thus the inner regularity of E with respect to ν is established.

On the other hand, in case $\nu(E) = \infty$, either $\nu(E\frown X_1)$ or $\nu(E\frown X_2)$ is infinite. For instance, suppose that $\nu(E\frown X_1) = \infty$. Then, $\mu_2(E\frown X_1)$ and consequently $\mu_1(E\frown X_1)$ are infinite (by Property (A)). On account of the inner regularity of μ_1, there exists a compact measurable set C such that $C \subseteqq E\frown X_1$ and $\mu_1(C) > M$ for an arbitrary $M > 0$, hence $\nu(C) \geqq \mu_1(C) > M$. Thus, the inner regularity of E with respect to ν is secured in this case, also.

Remark 2. We may place $\nu = \int f_1^{\frac{1}{p}} f_2^{\frac{1}{q}} d\mu \left(p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$ in place of $\nu = \int \sqrt{f_1 f_2}\, d\mu$ in Theorem 6, which is more general. In fact, we can proceed similarly by the so-called Hölder's inequality: $\nu(E)$

$\leqq(\mu_1(E))^{\frac{1}{p}}(\mu_2(E))^{\frac{1}{q}}$ $(E\in S)$ provided that $\mu_1(E)$ and $\mu_2(E)$ are finite (as a matter of course, $\nu(E)=\infty$ implies the infiniteness of $\mu_1(E)$ or $\mu_2(E)$).

We shall now state an example of a certain topological, measurable space on which $\mu_1=\int f_1 d\mu$ and $\mu_2=\int f_2 d\mu$ are both outer regular, and, nevertheless, $\nu=\int\sqrt{f_1 f_2}\,d\mu$ is not outer regular.

Example 3. Let X be $(0,\infty)$ with the topology induced by that of the real numbers. Let S be the σ-algebra generated by the class of all bounded, left closed, right open intervals in $(0,1)$, and $X_1=[1,\infty)$; and let μ be the Lebesgue measure thereon. Let us define the two measures, μ_1, μ_2, by the following identities:

$$f_1(x)=\begin{cases} x & (0<x<1) \\ \dfrac{1}{x^2} & (1\leqq x<\infty), \end{cases} \quad f_2(x)=\begin{cases} e^{\frac{1}{1-x}} & (0<x<1) \\ \dfrac{1}{x} & (1\leqq x<\infty), \end{cases} \quad \mu_1=\int f_1 d\mu,\ \mu_2=\int f_2 d\mu.$$

Then, $\sqrt{f_1 f_2}$ will be

$$\sqrt{f_1(x)f_2(x)}=\begin{cases} \sqrt{x\cdot e^{\frac{1}{1-x}}} & (0<x<1) \\ \dfrac{1}{x^{3/2}} & (1\leqq x<\infty). \end{cases}$$

Let $\nu=\int\sqrt{f_1 f_2}\,d\mu$.

Clearly μ_1 is outer regular by virtue of the strictly outer regularity of the Lebesgue measure μ and the boundedness of f_1. Next, μ_2 is also outer regular. Because, if $E\subseteqq(0,1)$, then $E=\bigcup_{n=1}^{\infty}E_n$, $E_n=E\frown\left(0,1-\dfrac{1}{n+1}\right)$, f_2 is bounded in $E_n(n=1,2,\cdots)$, and, secondly, if $E\supseteqq[1,\infty)$, then $\mu_2(E)=\infty$. On the other hand, $[1,\infty)$ is not outer regular with respect to ν, for, in spite of the finiteness of $\nu([1,\infty))$, $\nu(U)=\infty$ holds regarding every open measurable set U containing $[1,\infty)$.

In fact, in this example, the fact that $\mu_1([1,\infty))<\infty$ and $\mu_2([1,\infty))=\infty$ simultaneously implies the falsity of Property (A).

5. Irregular measures. Let μ_1, μ_2 be two measures such that a certain $E_0\in S$ be inner (outer) irregular with respect to both μ_1 and μ_2. Here the circumstances about the inner (outer) regularities of E_0 with respect to $\mu_1\smile\mu_2$ and $\mu_1\frown\mu_2$ are entirely different. The following theorem and examples will clarify the affairs above-mentioned.

Theorem 7. If a set $E_0\in S$ is inner (outer) irregular with respect to both μ_1 and μ_2, then E_0 is inner (outer) irregular with respect to $\mu_1\smile\mu_2$, too.

Proof. The case of "inner". We distinguish the four cases:

I. $\mu_1(E_0)<\infty$, $\mu_2(E_0)<\infty$. In this instance, $\nu(E_0)\leqq\mu_1(E_0)+\mu_2(E_0)$

$<\infty$ holds, but there exists a positive number k such that $\mu_1(E_0-C)>k$ for any compact measurable set C contained in E_0, hence $\nu(E_0-C) \geqq \mu_1(E_0-C)>k$.

II. $\mu_1(E_0)<\infty$, $\mu_2(E_0)=\infty$. Now, $\nu(E_0)\geqq\mu_2(E_0)=\infty$, and, on the other hand, there exists a positive number k such that $\mu_2(C)<k$ for any compact measurable set C contained in E_0, therefore $\nu(C)\leqq\mu_1(C) +\mu_2(C)<\mu_1(E_0)+k$.

III. $\mu_1(E_0)=\infty$, $\mu_2(E_0)<\infty$. Similar to Case II.

IV. $\mu_1(E_0)=\mu_2(E_0)=\infty$. In spite of the infiniteness of $\nu(E_0)$, the assertion that $\mu_1(C)<k_1$, $\mu_2(C)<k_2$ for any compact measurable set C contained in E_0 for some positive numbers k_1, k_2, implies $\nu(C)\leqq\mu_1(C) +\mu_2(C)<k_1+k_2$.

The case of "outer". Now, necessarily $\mu_1(E_0)<\infty$, $\mu_2(E_0)<\infty$, hence $\nu(E_0)<\infty$. However, there exists a positive number k such that $\mu_1(U-E_0)>k$ for any open measurable set U containing E_0, and accordingly $\nu(U-E_0)\geqq\mu_1(U-E_0)>k$.

The following examples show that Theorem 7 fails if we place $\mu_1\frown\mu_2$ in place of $\mu_1\smile\mu_2$.

Example 4. Let (X_1, S_1, m_1) be a topological measure space, and a set $E_1\in S_1$ be inner irregular with respect to m_1, and moreover, let (X_2, S_2, m_2) be also a topological measure space, and a set $E_2\in S_2$ be inner irregular with respect to m_2 provided that $X_1\frown X_2=\theta$. Consider now a topological measurable space (X, S) and two measures μ_1, μ_2 on S as follows:

$X=X_1\smile X_2$,

a set U in X is open if and only if $U=U_1\smile U_2$, U_1 being an open set in X_1 and U_2 an open set in X_2,

a set F in X is measurable (S) if and only if $F=F_1\smile F_2$, F_1 being measurable (S_1) and F_2 measurable $(S)_2$,

$\mu_1(F)=m_1(F\frown X_1)$, $\mu_2(F)=m_2(F\frown X_2)$ $(F\in S)$.

Then, $\nu=\mu_1\frown\mu_2\equiv 0$ and ν is trivially inner (outer) regular, but a set $E_0=E_1\smile E_2$ is surely inner irregular with respect to both μ_1 and μ_2. In order to explain the above statement, we shall first ascertain that a compact set in X is a union of a compact set in X_1 and a compact set in X_2. Now let C be a compact set in X. If we assume that $C\frown X_1\subseteq\bigcup_{\lambda\in\Lambda}U_\lambda$ where every U_λ is an open set in X_1, then $C=(C\frown X_1) \bigcup(C\frown X_2)\subseteq(\bigcup_{\lambda\in\Lambda}U_\lambda)\smile X_2$, and $U_\lambda(\lambda\in\Lambda)$ and X_2 are open in X, and accordingly $C\subseteq(\bigcup_{n=1}^{k}U_{\lambda_n})\smile X_2$, $\lambda_n\in\Lambda$ $(n=1, 2,\cdots, k)$.

Therefore, $C\frown X_1\subseteq\bigcup_{n=1}^{k}U_{\lambda_n}$ holds, and thus the compactness of $C\frown X_1$ is proved. Similarly, $C\frown X_2$ is also compact in X_2. Well, with respect to an arbitrary compact measurable set C in X contained in E_0, the identities $E_0-C=(E_1-(C\frown X_1))\smile(E_2-(C\frown X_2))$, $\mu_1(E_0-C) =\mu_1(E_1-(C\frown X_1))=m_1(E_1-(C\frown X_1))$ and $\mu_2(E_0-C)=\mu_2(E_2-(C\frown X_2))$

$=m_2(E_2-(C\frown X_2))$ imply the inner irregularity of E_0 with respect to both μ_1 and μ_2, provided that $\mu_1(E_0)=m_1(E_1)<\infty$, $\mu_2(E_0)=m_2(E_2)<\infty$. If, however, either one or both of $m_1(E_1)$ and $m_2(E_2)$ are infinite, the relations $\mu_1(C)=\mu_1(C\frown X_1)=m_1(C\frown X_1)$, $C\frown X_1\subseteq E_1$, $\mu_2(C)=\mu_2(C\frown X_2)$ $=m_2(C\frown X_2)$, $C\frown X_2\subseteq E_2$ will be of use for the same results.

Example 5. We have only to assume in Example 4 that E_1 and E_2 be outer irregular with respect to m_1 and m_2, respectively. Then, similarly to the above statements, E_0 will be outer irregular with respect to both μ_1 and μ_2.

References

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