80. *Normal Operators in Hilbert Spaces and Their Applications*

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(Comm. by K. KUNUGI, M.J.A., July 13, 1959)

In the present note, we wish to outline the following three problems for a compact or non-compact normal operator N in Hilbert space $\mathfrak{H}$ which is complete, separable, and infinite-dimensional:

1° the problem of finding characteristic elements and their corresponding characteristic values of N;

2° the problem of finding the multiplicities of characteristic values of N;

3° the problem of finding analytical properties of some normal operators associated with N.

The results which we shall give can be applied to linear non-homogeneous integral equations with normal kernels, but we will only give a few examples here.

The details will be shortly published in Memoirs of the Faculty of Education of Kumamoto University.

As a first step, we can easily establish the following lemmas:

Lemma 1. If N is a compact normal operator in $\mathfrak{H}$, then

(A) any non-null complex number different from all characteristic values of N belongs to the resolvent set;

(B) supposing that $\{\lambda_\nu\}_{\nu=1,2,\cdots}$ is the sequence of all characteristic values of N, arranged in an order such that $|\lambda_1|\geqq|\lambda_2|\geqq\cdots$, and denoting by E_ν the characteristic projector of N corresponding to λ_ν, $N=\sum_\nu \lambda_\nu E_\nu$ where the right-hand member converges uniformly to N in the case that $\{\lambda_\nu\}$ is an infinite sequence.

Lemma 2. Let N be a compact normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}$ and $\{E_\nu\}$ be the same symbols as those used in Lemma 1 respectively; and let $f^{(k)}$ be an arbitrary element of $\mathfrak{H}$ such that $E_\nu f^{(k)}=0$ for $\nu=1,2,\cdots,k-1$ and $E_k f^{(k)}\neq 0$. Then

$$|\lambda_k|=\lim_{n\to\infty}\|N^n f^{(k)}\|^{\frac{1}{n}}.$$

Lemma 2 here can be derived by a utilization of the expansion $N=\sum_\nu \lambda_\nu E_\nu$ in Lemma 1.

Put

$$g^{(k)}\equiv\frac{\sum_{\nu=k}^{p} E_\nu f^{(k)}}{\left\|\sum_{\nu=k}^{p} E_\nu f^{(k)}\right\|}\quad(1\leqq k\leqq p<\infty),\quad f_n\equiv(N^*N)^n f^{(k)},$$

$$g_n \equiv \frac{f_n}{\|f_n\|} = \frac{\sum_{\nu \geq k} |\lambda_\nu|^{2n} E_\nu f^{(k)}}{\sqrt{\sum_{\nu \geq k} |\lambda_\nu|^{4n} \|E_\nu f^{(k)}\|^2}}.$$

Then, by considering $\|g_n - g^{(k)}\|^2$ we can establish the following theorem:

Theorem 1. Let N be a compact normal operator in $\mathfrak{H}$; let f_0 be an arbitrary element such that $Nf_0 \neq 0$; and put

$$g_n = \frac{(N^*N)^n f_0}{\|(N^*N)^n f_0\|}.$$

Then there exists a unique element g such that $\lim_{n\to\infty} \|g_n - g\| = 0$ and it is a characteristic normalized element either of N or of N^*N; and accordingly, in the former case, (Ng, g) gives the corresponding characteristic value of N.

Furthermore a necessary and sufficient condition that the above strong limit g derived from every $f_0 \in \mathfrak{H}$ with $Nf_0 \neq 0$ be always a characteristic element of N is that all the characteristic values of N be mutually distinct in their absolute values; and if, hence, α is a suitable complex number such that the absolute values of all characteristic values of $N - \alpha I$ differ from each other and can be arranged in the monotone-decreasing sequence, the strong limit $g(\alpha)$ of $g_n(\alpha)$ defined by

$$g_n(\alpha) = \frac{[(N^* - \bar{\alpha} I)(N - \alpha I)]^n f_0}{\|[(N^* - \bar{\alpha} I)(N - \alpha I)]^n f_0\|},$$

where f_0 is an arbitrary element of $\mathfrak{H}$ such that $(N - \alpha I) f_0 \neq 0$, gives necessarily a characteristic normalized element of $N - \alpha I$ and accordingly of N, and the corresponding characteristic value of N is given by $(Ng(\alpha), g(\alpha))$.

By making use of the expansion $N - \lambda_k I = \sum_\nu (\lambda_\nu - \lambda_k) E_\nu$ and of the method of the proof of Theorem 1, we can derive

Theorem 2. Let N be a compact normal operator in $\mathfrak{H}$; let α and β be its characteristic values derived from an element f_0 with $Nf_0 \neq 0$ and the element $f_0' = (N - \alpha I) f_0$ respectively by the method of Theorem 1. Then $|\alpha| > |\beta|$.

By applying Lemma 2, we can yield

Theorem 3. If N is a compact normal operator in $\mathfrak{H}$, a necessary and sufficient condition that a non-null complex number α be a characteristic value of N is that there exist at least one pair of two elements f, g in $\mathfrak{H}$ such that

1° $(N - \alpha I) f = g$,

2° $\lim_{n\to\infty} \|N^n f\|^{\frac{1}{n}} > \lim_{n\to\infty} \|N^n g\|^{\frac{1}{n}}$.

Let λ be an arbitrary point in the resolvent set of the operator N, let $\{E(z)\}$ denote the complex spectral family of N, let $\{\lambda_\nu\}$ and $\{E_\nu\}$

have the same meanings as before, and let $\{\psi_n\}$ be a complete orthonormal system in $\mathfrak{H}$. Then we can verify with the help of (A) in Lemma 1 that

$$(N-\lambda I)^{-1}=\int_G \frac{1}{z-\lambda}dE(z)$$
$$=\sum_\nu \frac{E_\nu}{\lambda_\nu-\lambda}$$

where G denotes the complex plane. Furthermore the double-norm $N(E_\nu)=\sum_{n=1}^{\infty}\|E_\nu\psi_n\|^2$ associated with E_ν is independent of the complete orthonormal system $\{\psi_n\}$ used to define it.

By making use of these facts, the calculus of residues, and the Cauchy theorem on a regular function, we can give the following theorem:

Theorem 4. Let N be a compact normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}$ be the point spectrum of N; let ∂D_ν denote the boundary of a simply connected domain D_ν containing λ_ν such that zero and any non-null characteristic value $\lambda_\mu \neq \lambda_\nu$ of N are points exterior to the closure $\overline{D}_\nu$ of D_ν; and let $\{\psi_n\}$ be an arbitrary complete orthonormal system in $\mathfrak{H}$. Then the multiplicity m_ν of λ_ν is given by

$$m_\nu=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\left\|\int_{\partial D_\nu}(N-\lambda I)^{-1}\psi_n d\lambda\right\|^2.$$

We shall turn to non-compact normal operators of a particular type to which the results established above are applicable with a minor modification.

Theorem 5. Let N be a non-compact normal operator in $\mathfrak{H}$. If there exist a non-null complex number κ, a positive integer p and a complete orthonormal system $\{\psi_n\}$ such that $\sum_{n=1}^{\infty}\|(N-\kappa I)^p\psi_n\|^2<\infty$, then the multiplicity m_ν of any characteristic value λ_ν, not κ, of N is finite and given by

$$m_\nu=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\left\|\int_{\partial D_\nu}(N-\lambda I)^{-1}\psi_n d\lambda\right\|^2,$$

where ∂D_ν denote the boundary of a simply connected domain D_ν containing λ_ν such that κ and every characteristic value, not λ_ν, of N are points exterior to the closure $\overline{D}_\nu$ of D_ν; and furthermore Lemma 2 and Theorems 1, 2, 3 hold for $N-\kappa I$ instead of N.

Since, by virtue of the hypothesis concerning $N-\kappa I$, we can easily show that $N-\kappa I$ is a compact normal operator, it is obvious that the latter half of the above theorem is valid. In addition, by making use of the calculus of residues, the Cauchy theorem, and the fact that the resolvent set of N consists of all points on the complex plane except for $\lambda_1, \lambda_2, \cdots, \kappa$, we can prove the former half in the same way as Theorem 4 was proved.

Theorem 6. Let N be a compact normal operator or a non-compact normal operator satisfying the assumption of Theorem 5; let $\{\lambda_\nu\}$ be its point spectrum; let $F(z, \lambda)$ be a meromorphic function with complex parameter λ, which has $z=\lambda$ as its unique pole; let $F(N, \lambda)$ denote the operator defined by $F(N, \lambda)=\int_G F(z, \lambda)dE(z)$ where λ is an arbitrary point in the resolvent set of N and where G and $\{E(z)\}$ have such meanings as we have already described; let c be an arbitrarily given point in the resolvent set of N; let ∂D denote the rectifiable boundary of a simply connected domain D comprising completely the point spectrum $\{F(\lambda_\nu, c)\}$ of $F(N, c)$ in itself; and let $Q(\lambda)$ be a function holomorphic on the closure $\overline{D}$ of D. Then

$$\frac{1}{2\pi i}\int_{\partial D} Q(\lambda)[\lambda I-F(N, c)]^{-1}\,d\lambda=Q[F(N, c)],$$

where the complex curvilinear integration along ∂D is taken in the positive direction.

By the use of the expansion $[\lambda I-F(N, c)]^{-1}=\sum_\nu \frac{E_\nu}{\lambda-F(\lambda_\nu, c)}$, the calculus of residues, and the fact that even if $\{\lambda_\nu\}$ is a denumerably infinite set, $\sum_{\nu=n}^{\infty}\|E_\nu f\|^2$ tends to 0 for every $f\in\mathfrak{H}$ as $n\to\infty$, we can establish the above theorem.

Corollary 1. Let $F_\lambda(N, c)$ be the resolvent operator of $F(N, c)$ which is defined by $\left[I-\frac{F(N, c)}{\lambda}\right]^{-1}=I+\frac{F_\lambda(N, c)}{\lambda}$, $(\lambda\in\partial D)$. Then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{F_\lambda(N, c)}{\lambda}d\lambda=F(N, c).$$

These results established above are applicable to linear non-homogeneous integral equations in the function space $L_2(\Delta, \mu)$ in which Δ is a Lebesgue μ-measurable set of finite or infinite measure in Euclidean space R_m of dimension m.

Definition. If the operator N generated by a kernel $N(x, y)$ is compact, then, for simplicity of language, $N(x, y)$ is called to be compact.

Theorem 7. Let $N(x, y)$ be a compact normal kernel in $L_{22}(\Delta\times\Delta)$; let $T(x, y)$ denote $\int_\Delta \overline{N(z, x)}N(z, y)d\mu(z)$; let $T_n(x, y)$, $n=2, 3, \cdots$, be the iterated kernels of $T(x, y)$; and put $f_n(x, y)=\int_\Delta T_n(x, y)f(y)d\mu(y)$ for an arbitrarily given $f(x)\in L_2(\Delta, \mu)$ such that $\int_\Delta N(x, y)f(y)d\mu(y)\neq 0$ for almost every $x\in\Delta$, and

$$g_n(x)=\frac{f_n(x)}{\left\{\int_{\Delta}|f_n(x)|^2\,d\mu(x)\right\}^{\frac{1}{2}}}.$$

Then there exists the strong limit function $g(x)\in L_2(\Delta,\mu)$ given by

$$\lim_{n\to\infty}\int_{\Delta}|g_n(x)-g(x)|^2\,d\mu(x)=0$$

and it is a characteristic normalized function either of $N(x,y)$ or of $T(x,y)$; and hence, in the former case, that is, in the case where

$$\frac{\int_{\Delta}N(x,y)g(y)d\mu(y)}{g(x)}$$

remains constant for almost every $x\in\Delta$, the corresponding characteristic value of $N(x,y)$ is given by the constant value or by

$$\int_{\Delta\times\Delta}N(x,y)g(y)\overline{g(x)}d(\mu\times\mu).$$

This is a direct consequence of Theorem 1.

Theorem 8. A necessary and sufficient condition that a non-null complex number α be a characteristic value of the kernel $N(x,y)$ stated in Theorem 7 is that there exist at least one pair of functions $f(x), g(x)\in L_2(\Delta,\mu)$ such that $\int_{\Delta}N(x,y)f(y)d\mu(y)-\alpha f(x)=g(x)$ for almost every $x\in\Delta$ and such that

$$\lim_{n\to\infty}\left\{\int_{\Delta}\left|\int_{\Delta}N_n(x,y)f(y)d\mu(y)\right|^2 d\mu(x)\right\}^{\frac{1}{2n}}$$
$$>\lim_{n\to\infty}\left\{\int_{\Delta}\left|\int_{\Delta}N_n(x,y)g(y)d\mu(y)\right|^2 d\mu(x)\right\}^{\frac{1}{2n}}$$

where $N_n(x,y)$, $n=2,3,\cdots$, denote the iterated kernels of $N(x,y)$.

This result is an obvious consequence of Theorem 3.

Theorem 9. Let $N(x,y)$ be the kernel stated in Theorem 7; let $\{\psi_n(x)\}$ be an arbitrarily given complete orthonormal system in $L_2(\Delta,\mu)$; let ∂D_ν denote the boundary of a simply connected domain D_ν such that a non-null characteristic value λ_ν of $N(x,y)$ lies in D_ν, while any other point belonging to the spectrum of $N(x,y)$ does not lie on the closure $\overline{D}_\nu$ of D_ν; and let $f_n(x,\lambda)$ be the solution of the equation $\int_{\Delta}N(x,y)f(y)d\mu(y)-\lambda f(x)=\psi_n(x)$ for an arbitrary $\lambda\in\partial D$. Then the multiplicity m_ν of λ_ν is given by

$$m_\nu=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\int_{\Delta}\left|\int_{\partial D_\nu}f_n(x,\lambda)\,d\lambda\right|^2 d\mu(x).$$

This theorem follows immediately from Theorem 4.

Theorem 10. Let the integral operator N generated by a given kernel $N(x, y)$ be such an operator as was stated in Theorem 6; let $F(N, \lambda)$, c, $Q(\lambda)$ and ∂D have the same meanings as those described in the statement of Theorem 6; and let $F_c(x, y)$ denote the kernel generating $F(N, c)$. If $f(x, \lambda)$ is the solution of the equation $\lambda f(x, \lambda) - \int_{\Delta} F_c(x, y) f(y, \lambda) d\mu(y) = g(x)$ for an arbitrarily given $g(x) \in L_2(\Delta, \mu)$, then, for almost every $x \in \Delta$,

$$\frac{1}{2\pi i} \int_{\partial D} Q(\lambda) f(x, \lambda) d\lambda = [Q(F(N, c))g](x),$$

where the complex line integration along ∂D is taken in the positive direction.

This theorem can be verified readily from Theorem 6.

As a particular case of the above relation, we have

Corollary 2. Let $N(x, y)$ be the kernel stated in Theorem 7, let ∂D denote the rectifiable boundary of a simply connected domain D containing completely the spectrum of $N(x, y)$ in itself, and let $f(x, \lambda)$ be the solution of the equation $\lambda f(x, \lambda) - \int_{\Delta} N(x, y) f(y, \lambda) d\mu(y) = g(x)$. Then, for almost every $x \in \Delta$,

$$\frac{1}{2\pi i} \int_{\partial D} f(x, \lambda) d\lambda = g(x),$$

where the complex integration along ∂D is taken in the positive direction.