79. *On Fatou's Theorem*

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1. As one of the classical theorems in the theory of functions, the following Fatou's theorem is well known:

"If $f(z)$ is regular and bounded in the unit circle, then at almost all points of the unit circle the boundary values of $f(z)$ exist".

It seems to me that the proof, due to Carathéodory, is based on the fact that the boundary value is a differential coefficient of a function which satisfies "Lipschitz condition". In this paper we shall get into an argument so that the boundary value should be a differential coefficient of a function VBG_*.[1]

2. After this, we shall consider the function $f(z)$, one-valued regular in the unit circle: $|z|<1$. First of all, we pose the following condition (A):

(A) on the unit circle $C: |z|=1$, there exists a closed set N such that

(i) mes. $N=0$[2]

(ii) $\sup_{0\leq r<1}|f(re^{i\theta})|<\infty$ for $\theta\in C-N$.[3]

Proposition 1. *Under the condition* (A), *if we set*

$$F(\rho,\theta)=\int_{P_0}^{\theta} f(\rho e^{i\varphi})d\varphi,\quad \theta_0\notin N,\quad 0\leq\rho<1,$$

then for every $\theta\in C-N$ *there exists the limit:*

$$\lim_{\rho\to 1} F(\rho,\theta).$$

Proof. As is easily seen, we can set $f(0)=0$, and suppose that there exists a sequence of sets $\{E_n\}$, such that

(1°) $\sum_{n=1}^{\infty} E_n=C-N$,

(2°) if $\theta\in E_n$ then $\sup_{0\leq\rho<1}\left|\frac{f(\rho e^{i\theta})}{\rho e^{i\theta}}\right|\leq n_0+n$,

(3°) $\theta=0\in E_1$ ($\notin N$).

We shall set $A=\rho$, $B=\rho+\Delta\rho$, $C=\rho e^{i\theta}$, $D=(\rho+\Delta\rho)e^{i\theta}$, $(0\leq\rho<\rho+\Delta\rho<1)$ then

$$F(\rho+\Delta\rho,\theta)-F(\rho,\theta)=\int_0^{\theta} f\{(\rho+\Delta\rho)e^{i\varphi}\}d\varphi-\int_0^{\theta} f(\rho e^{i\varphi})d\varphi$$

$$=\int_B^D \frac{f(z)}{iz}dz-\int_A^C \frac{f(z)}{iz}dz,$$

1) Cf. S. Saks: Theory of the Integral.

2) mes. N means the measure of the set N.

3) $C-N=\{\theta:\ \theta\in C,\ \theta\notin N\}$.

as $f(z)/z$ is regular in $|z|<1$,

$$\int_A^B \frac{f(z)}{iz}dz+\int_B^D \frac{f(z)}{iz}dz+\int_D^C \frac{f(z)}{iz}dz+\int_C^A \frac{f(z)}{iz}dz=0,$$

therefore

$$F(\rho+\Delta\rho,\theta)-F(\rho,\theta)=-\int_A^B \frac{f(z)}{iz}dz+\int_C^D \frac{f(z)}{iz}dz,$$

where the curvilinear integrals should be calculated along the radius from A to B, from C to D, and along the arc from B to D, from C to A. If $\theta\in E_n$, by (2°)

$$|F(\rho+\Delta\rho,\theta)-F(\rho,\theta)|\leqq 2(n_0+n)\cdot\Delta\rho.$$

Therefore if $\theta\in E_n$, there exists uniformly $\lim_{\rho\to 1} F(\rho,\theta)$. Q.E.D.

If we define

$$F(\theta)=\begin{cases}\lim_{\rho\to 1} F(\rho,\theta) & \theta\in C-N\\ 0 & \theta\in N\end{cases}$$

the finite function on C is defined.

Next, we shall set the following condition (B):

(B) $\int_0^\theta F(r,\varphi)d\varphi$ are equi-absolutely continuous integrals, i.e. for every $\varepsilon>0$ there exists $\delta>0$, independently of r, such that for every non-overlapping intervals $\{I_k=[a_k,b_k]\}$, $(k=1,2,\cdots,k_0)$ the inequality $\sum_{k=1}^{k_0}|b_k-a_k|<\delta$ implies $\sum_{k=1}^{k_0}\left|\int_{a_k}^{b_k}F(r,\varphi)d\varphi\right|<\varepsilon$.

Proposition 2. *Under the conditions* (A), (B), $F(\theta)$ *is summable and*

$$(*)\qquad \lim_{\rho\to 1}\int_0^{2\pi}|F(\rho,\varphi)-F(\varphi)|\,d\varphi=0.$$

Proof. First we shall show, for every $\varepsilon>0$ there exists $\delta>0$, such that if E is a measurable set and mes. $E<\delta$ then independently of r we get

$$\int_E |F(r,\varphi)|\,d\varphi\leq\varepsilon.$$

In fact, from (B) we can select $\delta>0$, such that for every sequence of non-overlapping intervals $\{I_k\}$ which satisfies $\sum_{k=1}^{\infty}$ mes. $I_k<\delta$, the inequalities $\sum_{k=1}^{\infty}\left|\int_{I_k}F(r,\varphi)d\varphi\right|\leq\varepsilon/2$ hold independently of r. Let a measurable set E be mes. $E<\delta$. Then for every $\eta>0$ there exists a sequence of non-overlapping intervals $\{J_k\}$ such that $G=\sum_{k=1}^{\infty}J_k\supseteq E$, $\sum_{k=1}^{\infty}$ mes. $J_k<$ mes. $E+\eta$. Now we select η, $0<\eta<\mathrm{Min}\,[\varepsilon/2M_r,\ \delta-\text{mes. }E]$, where M_r denotes $\sup_{0\leq\varphi\leq 2\pi}|F(r,\varphi)|$. Then,

$$\left|\int_E F(r,\varphi)d\varphi\right|\leq\sum_{k=1}^{\infty}\left|\int_{J_k}F(r,\varphi)d\varphi\right|+\int_{G-E}|F(r,\varphi)|\,d\varphi\leq\varepsilon/2+M_r\cdot\varepsilon/2M_r=\varepsilon.$$

Next, let $F_1(r,\varphi)$, $F_2(r,\varphi)$ be the real and imaginary parts of $F(r,\varphi)$. If we apply the above fact, for $\varepsilon/4$, to $F(r,\varphi)$, there exists $\delta>0$ such that if E is a measurable set and mes. $E<\delta$, then independently of r,

$$\left|\int_E F(r,\varphi)d\varphi\right|\leq\varepsilon/4$$

hold. We shall denote for every r

$$E'_r=\{\varphi:\varphi\in E,\ F_1(r,\varphi)\geq 0\},\qquad E''_r=\{\varphi:\varphi\in E,\ F_1(r,\varphi)<0\},$$

then $E=E'_r+E''_r$ and

$$\int_{E'_r}|F_1(r,\varphi)|d\varphi=\int_{E'_r}F_1(r,\varphi)d\varphi=\left|\int_{E'_r}F_1(r,\varphi)d\varphi\right|\leq\left|\int_{E'_r}F(r,\varphi)d\varphi\right|\leq\varepsilon/4,$$

$$\int_{E''_r}|F_1(r,\varphi)|d\varphi=-\int_{E''_r}F_1(r,\varphi)d\varphi=\left|\int_{E''_r}F_1(r,\varphi)d\varphi\right|\leq\left|\int_{E''_r}F(r,\varphi)d\varphi\right|\leq\varepsilon/4.$$

Therefore $$\int_E|F_1(r,\varphi)|d\varphi\leq\varepsilon/4+\varepsilon/4=\varepsilon/2.$$

Similarly $\int_E|F_2(r,\varphi)|d\varphi\leq\varepsilon/2$, and finally $\int_E|F(r,\varphi)|d\varphi\leq\varepsilon$ hold independently of r.

In particular, there exists $\delta_1>0$ and if the interval I is of mes. $I<\delta_1$, then independently of r, we get $\int_I|F(r,\varphi)|d\varphi\leq 1$. On the other hand, $|F(r,\varphi)|$ tend to $|F(\varphi)|$ almost everywhere. Therefore by Fatou's lemma $|F(\theta)|$ is a summable function on I, consequently $F(\theta)$ is summable on C.

Given any $\varepsilon>0$, we can find $\delta>0$, such that mes. $E<\delta$ implies

$$\int_E|F(r,\varphi)|d\varphi<\varepsilon/3\qquad\text{(independently of } r)$$

and

$$\int_E|F(\varphi)|d\varphi<\varepsilon/3.$$

If we set $A_r(\sigma)=\{\varphi:|F(r,\varphi)-F(\varphi)|\geq\sigma\}$, $B_r(\sigma)=\{\varphi:|F(r,\varphi)-F(\varphi)|<\sigma\}$, for $0<\sigma<\varepsilon/6\pi$. There exists $\eta>0$ by Proposition 1, such that $0<1-r<\eta$ implies mes. $A_r(\sigma)<\delta$.

Then $$\int_0^{2\pi}|F(r,\varphi)-F(\varphi)|d\varphi\leq\int_{A_r(\sigma)}|F(r,\varphi)-F(\varphi)|d\varphi+2\pi\sigma$$
$$\leq\int_{A_r(\sigma)}|F(r,\varphi)|d\varphi+\int_{A_r(\sigma)}|F(\varphi)|d\varphi+\varepsilon/3.$$

If $1-r<\eta$ two terms of the last are less than $\varepsilon/3$. Therefore we get the result (*). Q.E.D.

We can see that the condition (B) is, in a sense, the best to get the result (*), i.e. the condition (B) is equivalent to the result (*).

3. *Proposition* 3. *Under the conditions* (A), (B), *and* $f(0)=0$, *we get*

$$f(re^{i\theta})=-\frac{1}{2\pi}\int_0^{2\pi}F(\varphi)\frac{d}{d\varphi}\Big(\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\Big)d\varphi \qquad (0\leq r<1).$$

Proof

$$f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{i\varphi})\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\varphi-\theta)+r^2}d\varphi \qquad (0\leq r<\rho<1)$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}F(\rho,\varphi)\frac{d}{d\varphi}\Big(\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\varphi-\theta)+r^2}\Big)d\varphi.$$

For brevity we shall write

$$P(r,\rho;\theta,\varphi)=\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\varphi-\theta)+r^2} \qquad (0\leq r<\rho<1),$$

then

$$f(re^{i\theta})+\frac{1}{2\pi}\int_0^{2\pi}F(\varphi)\frac{d}{d\varphi}P(r,1;\theta,\varphi)d\varphi$$

$$=-\frac{1}{2\pi}\Big[\int_0^{2\pi}\{F(\rho,\varphi)-F(\varphi)\}\frac{d}{d\varphi}P(r,\rho;\theta,\varphi)d\varphi$$

$$+\int_0^{2\pi}F(\varphi)\Big\{\frac{d}{d\varphi}P(r,\rho;\theta,\varphi)-\frac{d}{d\varphi}P(r,1;\theta,\varphi)\Big\}d\varphi\Big].$$

We denote these two integrals on the right by I_1, I_2, and select η as $0<\eta<\frac{1-r}{2}$. If $1-\rho<\eta$, then

$$\Big|\frac{d}{d\varphi}P(r,\rho;\theta,\varphi)\Big|\leq 2(1-r^2)\Big/\Big(\frac{1-r}{2}\Big)^4,$$

therefore

$$|I_1|\leq\Big\{2(1-r^2)\Big/\Big(\frac{1-r}{2}\Big)^4\Big\}\int_0^{2\pi}|F(\rho,\varphi)-F(\varphi)|\,d\varphi,$$

and $\rho\to 1$ implies $|I_1|\to 0$. Similarly

$$\Big|\frac{d}{d\varphi}P(r,\rho;\theta,\varphi)-\frac{d}{d\varphi}P(r,1;\theta,\varphi)\Big|\leq K\cdot(1-\rho), \quad 0<1-\rho<\eta$$

where K is a constant. If we set

$$\int_0^{2\pi}|F(\varphi)|\,d\varphi=M,$$

we know

$$|I_2|\leq K\cdot M(1-\rho),$$

and $\rho\to 1$ implies $|I_2|\to 0$.

Consequently we get $I_1+I_2=0$. Q.E.D.

Finally we set up rather complicated condition (C):

(C) there exists a sequence of the subsets of C, $\{G_n\}$, and a sequence of numbers $\{m_n\}$ such that

(i) $\sum_{n=1}^{\infty}G_n=C-N$

(ii) independently of ρ, $V_*(F(\rho, \theta), G_n) \leq m_n$.[4]

Proposition 4. *Under the conditions* (A), (C), $F(\theta)$ *is* VBG_* *on* $C-N$, *therefore* $F(\theta)$ *is derivable at almost all points of* C.

Proof. As N is a closed set, there exists a sequence of open intervals $\{I_k\}$ such that $C-N=\sum_{k=1}^{\infty} I_k$. We set $G_{n,k}=G_n \cdot I_k$ $(n, k=1, 2, \cdots)$ then $C-N=\sum_{n,k=1}^{\infty} G_{n,k}$. We shall show $F(\theta)$ is VB_* on each set $G_{n,k}$. Let $J_k=[\alpha_k, \beta_k]$, $k=1, 2, \cdots, k_0$ be non-overlapping subintervals of I_k whose end-points belong to G_n. For every J_k we select arbitrary subinterval of J_k, that is $L_k=[\gamma_k, \delta_k]$. For $\gamma_k, \delta_k \notin N$, if we select ρ sufficiently near to 1

$$|F(\gamma_k)-F(\rho, \gamma_k)|+|F(\delta_k)-F(\rho,\delta_k)|<1/k_0 \quad (k=1, 2, \cdots, k_0),$$

therefore

$$\sum_{k=1}^{k_0} |F(\gamma_k)-F(\delta_k)| \leq \sum_{k=1}^{k_0} \{|F(\gamma_k)-F(\rho, \gamma_k)|+|F(\delta_k)-F(\rho, \delta_k)|\}$$
$$+\sum_{k=1}^{k_0} |F(\rho, \gamma_k)-F(\rho, \delta_k)| \leq \sum_{k=1}^{k_0} O(F(\rho, \theta); J_k)^{5)}+1 \leq m_n+1.$$

Then we get
$$\sum_{k=1}^{k_0} O(F(\theta); J_k) \leq m_n+1.$$

As $\{J_k\}$ is arbitrary, we know

$$V_*(F(\theta); G_{n,k}) \leq m_n+1<\infty. \qquad \text{Q.E.D.}$$

4. If a function $f(z)$, defined in the unit circle, has the following property we shall call "$f(z)$ has Fatou's property": on the unit circle C there exists a set N whose measure is zero, and if $\theta \in C-N$ then

$$\lim_{z \to e^{i\theta}} f(z)$$

exists, where $z \to e^{i\theta}$ means that z converges to $e^{i\theta}$ non-tangentially to C.

Now we can state the main result in the following theorem.

Theorem. If $f(z)$ *satisfies the conditions* (A), (B) and (C), *then* $f(z)$ *has Fatou's property.*

By the aid of Propositions 3, 4, the proof of above theorem could be carried out quite similarly to the classical Carathéodory's proof.

For example, a short verification would make clear that the function

$$\sum_{n=2}^{\infty} \frac{z^n}{\log n}$$

satisfies the conditions (A), (B) and (C).

4) Cf. S. Saks: Loc. cit. In this paper, we shall say that a complex-valued function is of bounded variation on a set E, if its real and imaginary parts are of bounded variation on E.

5) $O(F(\theta); I)=\sup_J |F(J)|$ where J is a subinterval of I.