75. On Ring Homomorphisms of a Ring of Continuous Functions. II

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Anderson and Blair [1] have investigated representations of certain rings as subalgebras of $C(X)$.[1] In this paper, we shall in §1 also consider such representations of certain rings and we shall improve Theorems 2.2 and 3.2 in [1] using results obtained in [2, 3]. From results in §1, we obtain in §2 new characterizations of locally Q-complete spaces, Q-spaces, locally compact spaces and compact spaces.

Let R be a ring of all real numbers. A subset A of $C(X)$ is said, according to [1], to be *weakly pseudoregular* if X has a subbase $\mathfrak{U}$ of open sets such that for any $U\in\mathfrak{U}$ and $x\in U$ there are an $\alpha>0$ (in R) and an f in A such that $|f(x)-f(y)|>\alpha$ for $y\notin U$. A is *pseudoregular*[2] if for any $x\in X$ and any open neighborhood U of x, there is an $f\in A$ such that $f(x)=0$ and $f(y)\geq 1$ for $y\notin U$. An element f in A is said to be *strictly positive* if there exists an $\alpha>0$ (in R) such that $f(x)\geq\alpha$ for every $x\in X$. Next suppose that A is an arbitrary algebra over R. A maximal ideal M of A is said to be *real* if the residue class algebra A/M is isomorphic to R. $\mathfrak{R}_A$ denotes the totality of real maximal ideals of A. An element f in A is said to be *strictly positive* if there exists $\alpha>0$ (in R) such that $M(f)\geq\alpha$ for every $M\in\mathfrak{R}_A$ where $M(f)=f$ mod. M. Let us put $S(f)=\{M(f);\ M\in\mathfrak{R}_A\}$ which is called a *spectrum* of f. If A is a subset of $C(X)$, and for any $M\in\mathfrak{R}_A$, there is a unique point x in X such that $M=M_x=\{f; f(x)=0\}$ then A is said to be *point-determining*; in other words, A has the property (H^*) in [3], that is, any ring homomorphism φ of A onto R is a point ring homomorphism φ_x and $x\neq y$ implies $\varphi_x\neq\varphi_y$.

1. Now suppose that A is a ring such that $\mathfrak{R}_A\neq 0$ and $\bigcap_{M\in\mathfrak{R}_A} M=\theta$ (written $\frown\mathfrak{R}_A=\theta$). We define a function f^* an $\mathfrak{R}_A$ by $f^*(M)=M(f)$, moreover, introduce a weak topology on $\mathfrak{R}_A$, that is, we take as a subbase of open sets of $\mathfrak{R}_A$, $\mathfrak{U}=\{U_M(f,\varepsilon);\ f\in A, \varepsilon\in R, \varepsilon>0\}$ where $U_M(f,\varepsilon)=\{N;\ |M(f)-N(f)|<\varepsilon, N\in\mathfrak{R}_A\}$. Then, by [1, Theorem 2.1], for any given X, a weakly pseudoregular point-determining subring A of $C(X)$

1) In the following, X is always a completely regular T_1-space and other terminologies used here, for instance $C(X)$, ring homomorphisms and local Q-completeness, are the same as in [2, 3].

2) The definition of pseudoregular in [1] requires moreover that A contains a constant function e which takes value 1 on X.

characterizes X, i.e. X is homeomorphic to $\mathfrak{R}_A$ with the weak topology. Moreover, by [1, Theorem 2.2], and arbitrary ring A is isomorphic to a weakly pseudoregular point-determining subring A^* of $C(X)$, for some topologically unique completely regular space X, if and only if $\frown\mathfrak{R}_A=\theta$. In this case X is a space $\mathfrak{R}_A$ with the weak topology and A is isomorphic to A^* by the correspondence $f\to f^*$.

LEMMA 1. *A subalgebra (=a linear subring) of $C(X)$ generates a structure of X if and only if A is weakly pseudoregular.*

Proof is obvious.

The existence of weakly pseudoregular point-determining subring A of $C(X)$ does not necessarily imply that X is a Q-space (see Example 1 in [1]). Moreover, even if A is a subalgebra, X is not necessarily a Q-space. Such an example is given by $C_B(X)$ where X is locally Q-complete and B is a compact subset of $\beta X-X$ containing $\nu X-X$ (see Theorem 1 in [2]). But by Theorem 1 and Corollary 3 in [2] and Theorem 2.2 in [1] we have

THEOREM 1. *An algebra A is isomorphic to a weakly pseudoregular point-determining subalgebra of $C(X)$ for some topologically unique locally Q-complete space X if and only if $\frown\mathfrak{R}_A=\theta$.*

Next we improve Lemma 3.1 in [1] using the same method used as in the proof of (2) in [3].

LEMMA 2. *If A is a point-determining weakly pseudoregular subalgebra of $C(X)$ which has a strictly positive function, then X is a Q-space.*

Proof. Suppose that $\nu X-X\neq\theta$. For any point $y\in\nu X-X$ $\varphi_y(f)=\tilde{f}(y)$ for $f\in A$ is a ring homomorphism of A into R where $\tilde{f}$ denotes a continuous extension of f over νX. If $\varphi_y(A)\neq 0$ then, since A is linear, $\varphi_y(A)=R$. Therefore there is a point x in X such that $\varphi_x=\varphi_y$. But this leads to a contradiction by the same method used as in (2) in [3]. Thus we have that $\varphi_y(A)=0$ for any point y in $\nu X-X$. Moreover, this means that X is open in νX and that any function in A vanishes on $\nu X-X$. Now let $x\in X$ and U be an open subset containing x such that $\bar{U}$ (in $\nu X)\frown(\nu X-X)=\theta$. By the assumption, there is a strictly positive function $f\geq\alpha$ for some $\alpha>0$. Then $\tilde{f}\geq\alpha$ on $\nu X-X$. This is a contradiction. Thus X must be a Q-space.

LEMMA 3. *If A is pseudoregular subring of $C(X)$, then A contains a strictly positive function.*

Proof. Let x and y be two distinct points and let U and V be disjoint open neighborhoods of x and y respectively. By the pseudoregularity, there are f and g such that $f(x)=0$, $f(X-U)\geq 1$ and $g(y)=0$, $g(X-V)\geq 1$. Then f^2+g^2 belongs to A and takes a value not smaller than 1 on X, that is, A contains a strictly positive function.

Thus we have

THEOREM 2. *An algebra A is isomorphic to a weakly pseudoregular point-determining subalgera of $C(X)$, with some strictly positive function, for some topologically unique Q-space if and only if $\frown\mathfrak{R}_A=\theta$ and A has a strictly positive function.*

From Lemma 3 and Theorem 2 we have

COROLLARY. *If $C(X)$ has a pseudoregular point-determining subalgebra, then X is a Q-space.*

2. Let $PA(X)$ be the totality of weakly pseudoregular point-determining subalgebras of $C(X)$. Since $\frown\mathfrak{R}_A=\theta$ for any $A\in PA(X)$ and A is weakly pseudoregular, by Theorem 1 X must be locally Q-complete. If A contains a strictly positive function, then by Theorem 2 X must be a Q-space. Conversely if X is a Q-space, it is well known that $C(X)$ belongs to $PA(X)$ and contains, of course, strictly positive functions.

Next suppose that X is compact and $A\in PA(X)$. For any point x, there is a function $f_x\in A$ such that $f_x(x)\neq 0$. Then $\mathfrak{U}=\{U(x,f_x^2,\varepsilon_x);\ x\in X\}$ becomes an open covering of X where $U(x,f_x^2,\varepsilon_x)=\{y;\ |f_x^2(x)-f_x^2(y)|<\varepsilon_x\}$ and $f_x^2(x)>2\varepsilon_x>0$. By the compactness of X, there is a subcovering $\{U(x_i,f_{xi}^2,\varepsilon_{xi});\ i=1,2,\cdots,n\}$ of $\mathfrak{U}$. Then $g=\sum_{i=1}^{n}\frac{1}{\varepsilon}f_{xi}^2$ belongs to A and $g\geq 1$ on X where $\varepsilon=\min(\varepsilon_{x_1},\cdots,\varepsilon_{x_n})$. Therefore any A belonging to $PA(X)$ has a strictly positive function. Conversely, suppose that X is not compact. X is locally Q-complete by Theorem 1. Let B be a compact subset contained in $\beta X-X$ which i) in case X is not a Q-space, B is compact subset containing $\nu X-X$, ii) in case X is a Q-space, B is any compact subset. Then by [2, Theorem 1] $C_B(X)$ is a point-determining subalgebra of $C(X)$, and any function f in $C_B(X)$ has a continuous extension f over B on which $f=0$. This implies that $C_B(X)$ has no strictly positive functions. Thus we have the following

THEOREM 3. *Let X be a completely regular T_1-space; then* 1) *X is locally Q-complete if and only if $PA(X)\neq\theta$,* 2) *X is a Q-space if and only if $PA(X)\neq\theta$ and some subalgebra belonging to $PA(X)$ has a strictly positive function,* 3) *X is compact if and only if $PA(X)\neq\theta$ and every subalgebra belonging to $PA(X)$ contains a strictly positive function.*

REMARK. Suppose that A belongs to $PA(X)$ and $S(f)$ is bounded for every $f\in A$. Then, in the arguments in §§1–2, we can replace νX by βX and we have the results that local Q-completeness and Q-space are replaced, in Theorems 1, 2 and Corollary, local compactness and compact space respectively.

Let $PI(X)$ be the totality of ideals of $C(X)$ which belongs to

$PA(X)$. If $PA(X)\neq\theta$, i.e. X is locally Q-complete, then $PI(A)\neq\theta$. For, i) in case X is a Q-space, $C(X)\in PI(X)$, ii) in case X is not a Q-space, $C_B(X)\in PI(A)$ where $B=(\nu X-X)^\beta$.

Suppose that A belongs to $PI(A)$ and contains a strictly positive function, then it is easy to see that $A=C(X)$ because A is an ideal of $C(X)$. This shows that if X is compact, then $PI(X)$ consists of only one element $C(X)$. If X is not compact and some $A\in PI(X)$ contains a strictly positive function, then $A=C(X)$ and this implies that X is a Q-space. If X is a Q-space, $C_B(X)$ belongs to $PI(A)$ where $B=\{b\}$ and b is any point in $\beta X-X$. On the other hand, a pseudo-compact Q-space is compact and if Y is not pseudo-compact, $\overline{\overline{(\beta Y-Y)}}\geq 2^{\aleph_0}$ where $\overline{\overline{Y}}$ denotes the cardinal number of Y and $\aleph_0$ denotes the cardinal number of a set of all integers. Therefore if X is a Q-space and not compact, $\overline{\overline{PI(X)}}\geq 2^{\aleph_0}$ and $C(X)\in PI(X)$.

Next we shall consider the converse. Suppose that $PI(X)$ consists of only one element $C(X)$. By the result obtained by Hewitt, X is a Q-space. If X is not compact, $\overline{\overline{PI(X)}}\geq 2^{\aleph_0}$, and hence X must be compact. Next suppose that $C(X)\in PI(X)$ and $\overline{\overline{PI(X)}}\geq 2^{\aleph_0}$. Then X is a Q-space and X is not pseudo-compact, thus X is a Q-space and not compact.

If $PI(X)\ni A$ and $A\neq C(X)$, then $Z(f)\neq\theta$ for any $f\in A$, and we have $\Delta(A)=\bigcap_{f\in A}Z(f)^\beta\neq\theta$ because A is an ideal of $C(X)$. Since A is point-determining, it is obvious that $\Delta(A)\supset(\nu X-X)^\beta$ (if there exists). We shall prove that if X is locally compact, $C_k(X)$ is the smallest ideal belonging to $PI(X)$. To prove this, it suffices to show that any $f\in C_k(X)$ is contained in A for any $A\in PI(X)$. Since the support F of f is compact and A is point-determining, there is a function g_x in A for each $x\in F$ which is positive on some open neighborhood of x. By the compactness of F, there is a function g in A which takes value greater than 1 on F because A is an ideal. Then $h=g^2/\max(g^2, 1)$ belongs to A and $h(F)=1$. By the method of construction of h, we have $hf=f$, and hence $f\in A$.

Conversely suppose that X is not locally compact and A_0 is the smallest ideal belonging to $PI(X)$. If $A_1, A_2\in PI(X)$ and $A_1\subset A_2$, then it is obvious that $\Delta(A_1)\supset\Delta(A_2)$. Thus for any $A\in PI(X)$, we have $\Delta(A_0)\supset\Delta(A)$ and $\Delta(A_0)$ is a compact subset in $\beta X-X$. Since $\beta X-X$ is not compact, there is a point b in $(\beta X-X)-\Delta(A_0)$ and $C_B(X)$ is not contained in A_0 where $B=\Delta(A_0)\cup\{b\}$. Thus we have

THEOREM 4. *Let X be a completely regular T_1-space.*

1) *X is locally Q-complete if and only if $PI(X)\neq\theta$,*

2) *X is compact if and only if $PI(X)$ consists of only one element $C(X)$,*

3) *X is a Q-space but not compact if and only if* $PI(X) \ni C(X)$ *and* $\overline{\overline{PI(X)}} \geq 2^{\aleph_0}$,
4) *X is locally compact if and only if* $PI(X)$ *has the smallest ideal.*

References

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