74. On Compact Semirings

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1. *Introduction.* In this paper we generalize to the infinite case our theorem that *a finite semiring without zeroid is a ring* [1]. We prove the natural extension that *a compact semiring without zeroid is a ring.* As a by-product, we obtain a generalization for the commutative case of Numakura's theorem that *a compact semigroup satisfying the cancellation law is a group* [4] to *a compact abelian semigroup without zeroid is a group.*

I. Kaplansky [2] has given structure theorems for compact rings. He proved that *a compact semi-simple ring is isomorphic and homeomorphic to a Cartesian direct sum of finite simple rings* [2]. Hence, this structure theorem remains true for a compact semi-simple semiring.

Only semirings with commutative addition and a zero, in the sense of Vandiver and Weaver [5], are considered. This paper has benefited materially from discussion with H. Zassenhaus of the University of Notre Dame.

2 *Quotient spaces. Definition* 1. A topological semiring is a semiring S together with a Hausdorff topology on S under which the semiring operations are continuous. Since the zeroid of a semiring will play an important role in what follows, we repeat its definition.

Definition 2. The zeroid $Z(S)$ of a semiring S is the set of elements z of S for which the equation $z+x=x$ is solvable in S.

In a previous paper [1] we defined two elements i_1, i_2 of a semiring S to be equivalent if the equation $i_1+x=i_2+x$ is solvable in S. These equivalence classes i^* represented by $i\in S$ form a semiring S^* with cancellation law of addition, according to the laws $i_1^*+i_2^*=(i_1+i_2)^*$, $i_1^*i_2^*=(i_1i_2)^*$. S^* is then a halfring [6]. The zeroid consists of all elements z of S for which $z^*=0$, i.e. the zeroid of S is the inverse image of the O-element of S^* under the homeomorphism $i\to i^*$ of S onto S^*.

We introduce in S^* the quotient topology, that is the largest topology for S^* such that the function $i\to i^*$ is a continuous mapping of S onto S^*. We assume that S is a compact space. Then S^* is also compact space, for the function $i\to i^*$ is a continuous mapping of S onto S^* [3].

LEMMA 1. *The compact space S^* is Hausdorff.*

Proof. We recall the following theorems: *Let X be a topological* *space, let $\mathfrak{D}$ be an upper semi-continuous decomposition of X whose members are compact and let $\mathfrak{D}$ have the quotient topology. Then $\mathfrak{D}$ is Hausdorff, provided X is Hausdorff* [3].

A decomposition $\mathfrak{D}$ of a topological space X is upper semi-continuous if and only if the projection P of X onto $\mathfrak{D}$ is closed [3].

We prove that the projection (quotient map) $i \to i^*$ is closed, that is the image of each closed set is closed. Let A be a closed subset of S and A^* its image. We show that A^* is closed subset of S^*. Since S^* is a quotient space of S this is equivalent to proving that the set $\hat{A}$ of all elements y of S, such that $y^* \in A^*$, is a closed subset of S. Since a compact subset of a Hausdorff space is closed, it is sufficient to prove that $\hat{A}$ is a compact subset of S. We recall that *a topological space X is compact if and only if each net in X has a subnet which converges to some point of X* [3]. Hence, we wish to show that the net $\{y_n\}$ in $\hat{A}$ has a subnet which converges to some point of $\hat{A}$. There exist x_n, such that $y_n^* = x_n^*$, that is $y_n + z_n = x_n + z_n$, $x_n \in A$ and $z_n \in S$. Since A is a closed subset of S, it is a compact subset of S. Hence, the net $\{x_n\}$ has a subnet $\{x_{\nu_n}\}$ which converges to some point x of A. Similarly the net $\{z_{\nu_n}\}$ possesses a subnet $\{z_{\rho_n}\}$ which converges to some point z of S. Hence, there exists a convergent subnet $\{y_{\sigma_n}\}$ of the net $\{y_{\rho_n}\}$ such that $y_{\sigma_n} + z_{\sigma_n} = x_{\sigma_n} + z_{\sigma_n}$, where $\lim x_{\sigma_n} = x$ and $\lim z_{\sigma_n} = z$. Because of the continuity of addition, $\lim y_{\sigma_n} + \lim z_{\sigma_n} = \lim x_{\sigma_n} + \lim z_{\sigma_n}$, $y + z = x + z$ and $y \in \hat{A}$. This implies that $\hat{A}$ is a compact subset of the Hausdorff space S and consequently a closed subset of S. The mapping $i \to i^*$ is upper semi-continuous.

Since $\{i\}$ is closed, S being a T_1-space, it follows that its image $\{i^*\}$ is also closed in S^* and the inverse image of $\{i^*\}$ is a closed subset of S and therefore a compact subset of S. The members of the decomposition of S are compact. Since S is Hausdorff, then also S^* is Hausdorff.

LEMMA 2. *The compact halfring S^* is a compact ring.*

Proof. The additive semigroup of S^* is a compact semigroup satisfying the cancellation law and hence is group by Numakura's theorem [4]. S^* is a compact ring.

LEMMA 3. *If S is a compact semiring without zeroid then S is a compact ring.*

Proof. Lemma 2 states that S^* is a ring. Hence, for any $x \in S$, $x^* + y^* = 0^*$ is soluble in S^* and $(x+y)^* = 0^*$. Since the zeroid $Z(S) = 0$, this implies that $x + y = 0$ and S is a ring.

As an immediate consequence of Lemma 3, we have

THEOREM 1. *A compact semimodule without zeroid is a module.*

This theorem is a generalization for the commutative case of

Numakura's theorem [4], stated in the introduction.

If S is semi-simple, its semiradical is zero. In our previous paper [1], we showed that the zeroid $Z(S)$ is a two-sided ideal contained in the semiradical. Hence $Z(S)=0$ and we have the Kaplansky result [2] for semirings.

THEOREM 2. *A compact semi-simple semiring is isomorphic and homeomorphic to a Cartesian direct sum of finite simple rings.*

References

[1] S. Bourne and H. Zassenhaus: On the semiradical of a semiring, Proc. Nat. Acad. Sci., U. S. A., **44**, 907–914 (1958).

[2] I. Kaplansky: Topological rings, Am. Jour. Math., **69**, 153–183 (1947).

[3] J. L. Kelley: General Topology, New York, Van Nostrand (1955).

[4] K. Numakura: On bicompact semigroups, Math. Jour. Okayama Univ., **1**, 102 (1952).

[5] H. S. Vandiver and M. W. Weaver: A development of associative algebra and an algebraic theory of numbers IV, Math. Mag., **30**, 6 (1956).

[6] H. Zassenhaus: The Theory of Groups, 2nd ed., New York, Chelsea, 95 (1958).