73. On the Unique Factorization Theorem in Regular Local Rings

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Recently Auslander and Buchsbaum [3] have proved that every regular local ring is a unique factorization ring. This proof depends upon the following result of Nagata [1]: *If every regular local ring of dimension 3 is a unique factorization ring, then so is every regular local ring of any dimension* (see [1, pp. 411–413]).

This theorem was proved independently by Zariski [2].

Nagata proved this theorem by using homological method and ideas. The purpose of this paper is to prove anew this theorem by a purely ideal-theoretic method in a simpler way than in [1] and [2].

Let $\mathfrak{O}$ be an n dimensional regular local ring.

Let $\mathfrak{m}=\mathfrak{O}u_1+\mathfrak{O}u_2+\cdots+\mathfrak{O}u_n$ be the maximal ideal of $\mathfrak{O}$, and $\mathfrak{O}'=\mathfrak{O}[X_1, X_2, \cdots, X_n]$ be the polynomial ring over $\mathfrak{O}$. Then $\mathfrak{m}'=\mathfrak{m}[X_1, X_2, \cdots, X_n]$ is a prime ideal of $\mathfrak{O}'$. Let $\mathfrak{O}^*$ be the quotient ring of $\mathfrak{O}'$ with respect to $\mathfrak{m}'$, then $\mathfrak{O}^*$ will be n dimensional regular local ring, and $\mathfrak{m}^*=\mathfrak{O}^*u_1+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$ will be the maximal ideal of $\mathfrak{O}^*$. In the following, we shall use $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{q}$, etc. to denote ideals in $\mathfrak{O}$, and $\mathfrak{a}^*$, $\mathfrak{b}^*$, $\mathfrak{p}^*$, $\mathfrak{q}^*$, etc. to denote ideals in $\mathfrak{O}^*$.

We note the following well-known lemma without proof (see, for example, [4]).

Lemma 1. *We have*

(i) $\mathfrak{O}\frown\mathfrak{O}^*\mathfrak{a}=\mathfrak{a}$.

(ii) *If* $\mathfrak{p}$ *is a prime ideal in* $\mathfrak{O}$, *then so is* $\mathfrak{O}^*\mathfrak{p}$ *in* $\mathfrak{O}^*$, *and if* $\mathfrak{q}$ *is* $\mathfrak{p}$*-primary, then* $\mathfrak{O}^*\mathfrak{q}$ *is* $\mathfrak{O}^*\mathfrak{p}$*-primary. Moreover rank* $\mathfrak{p}=$*rank* $\mathfrak{O}^*\mathfrak{p}$.

A less familiar lemma is:

Lemma 2. *Let* $v^*=u_1X_1+u_2X_2+\cdots+u_nX_n$, *then* v^* *is an element of a minimal base of* $\mathfrak{m}^*$. *Moreover,* $\mathfrak{O}^*\mathfrak{a}\ni v^*$ *holds if and only if* $\mathfrak{a}=\mathfrak{m}$.

Proof. From $\mathfrak{m}^*=\mathfrak{O}^*u_1+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$ follows the equation $\mathfrak{m}^*=\mathfrak{O}^*v^*+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$. Therefore v^* is an element of a minimal base of $\mathfrak{m}^*$.

Since every element of $\mathfrak{O}^*\mathfrak{a}$ can be expressed in the form $P(x)/Q(x)$, $P(x)\in\mathfrak{a}[X_1, X_2, \cdots, X_n]$, $Q(x)\notin\mathfrak{m}[X_1, X_2, \cdots, X_n]$, $\mathfrak{O}^*\mathfrak{a}\ni v^*$ implies that $\mathfrak{a}[X_1, X_2, \cdots, X_n]\ni v^*$, this means $\mathfrak{a}\ni u_1, u_2, \cdots, u_n$, and thereby completes the proof.

Now, let φ be a natural homomorphism of $\mathfrak{O}^*$ onto the regular local ring $\bar{\mathfrak{O}}=\mathfrak{O}^*/\mathfrak{O}^*v^*$ of dimension $n-1$.

Lemma 3. *Let $\mathfrak{O}$ be a regular local ring of dimension $n\geqq 3$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathfrak{O}$ with a condition rank $\mathfrak{a}=$rank $\mathfrak{b}=1$. Then, there exists a minimal prime ideal in $\mathfrak{O}$ belonging to $\mathfrak{a}$ and $\mathfrak{b}$, if and only if there exists a minimal prime ideal in $\bar{\mathfrak{O}}$ belonging to $\varphi(\mathfrak{O}^*\mathfrak{a})$ and $\varphi(\mathfrak{O}^*\mathfrak{b})$.*

Proof. Necessity is evident. Suppose that there exists a minimal prime ideal $\bar{\mathfrak{p}}$ which belongs to $\varphi(\mathfrak{O}^*\mathfrak{a})$ and $\varphi(\mathfrak{O}^*\mathfrak{b})$. From the assumption rank $\bar{\mathfrak{p}}=1$ follows rank $\varphi^{-1}(\bar{\mathfrak{p}})=2$. And we have $\varphi^{-1}(\bar{\mathfrak{p}})\supset\mathfrak{O}^*\mathfrak{a}$, $\mathfrak{O}^*\mathfrak{b}$. On the other hand, we have $\varphi^{-1}(\bar{\mathfrak{p}})\ni v^*$, this implies that rank $\mathfrak{O}\frown\varphi^{-1}(\bar{\mathfrak{p}})=1$, from Lemma 2. This means that there exists a minimal prime ideal in $\mathfrak{O}$ which belongs to $\mathfrak{a}$ and $\mathfrak{b}$.

Theorem. If every regular local ring of dimension 3 *is a unique factorization ring, then so is every regular local ring of any dimension.*

Proof. If dim $\mathfrak{O}=1$ or 2, it is easy to prove that $\mathfrak{O}$ is a unique factorization ring (see, for example, [1, Th. 4, p. 410]).

Therefore, for the purpose of the proof, we may assume that dim $\mathfrak{O}>3$, and may assume that every regular local ring of dimension less than dim $\mathfrak{O}$ is a unique factorization ring. Let $\mathfrak{p}$ be a prime ideal of rank 1 in $\mathfrak{O}$. Since $\mathfrak{p}\not\subseteq\mathfrak{p}^{(2)}+\mathfrak{p}\cdot\mathfrak{m}$ (where $\mathfrak{p}^{(2)}$ is the "symbolic square" of $\mathfrak{p}$, i.e. the $\mathfrak{p}$-primary component of $\mathfrak{p}^2$), there exists an element p_1 of $\mathfrak{p}$ such that $p_1\notin\mathfrak{p}^{(2)}$ and $p_1\notin\mathfrak{p}\cdot\mathfrak{m}$. Assume that $\mathfrak{p}\neq\mathfrak{O}p_1$. We shall show that this implies a contradiction. It is well known that this completes the proof (see, for example, [1, Lemma 1, p. 408]).

Since $p_1\notin\mathfrak{p}^{(2)}$, we have $\mathfrak{O}p_1=\mathfrak{p}\frown\mathfrak{a}$, where $\mathfrak{a}$ is unmixed, of rank 1 and not contained in $\mathfrak{p}$. Since $\mathfrak{a}:\mathfrak{p}=\mathfrak{a}$, there exists an element p_2 of $\mathfrak{p}$ such that $\mathfrak{a}:\mathfrak{O}p_2=\mathfrak{a}$. By assumption, $\bar{\mathfrak{O}}(=\varphi(\mathfrak{O}^*))$ is a unique factorization ring, consequently we have $\varphi(p_1)=\bar{g}\bar{a}$, where $\bar{g}$ and $\bar{a}$ are such elements of $\bar{\mathfrak{O}}$ that $\mathfrak{O}^*\mathfrak{p}\subset\varphi^{-1}(\bar{\mathfrak{O}}\bar{g})$, $\mathfrak{O}^*\mathfrak{a}\subset\varphi^{-1}(\bar{\mathfrak{O}}\bar{a})$. By Lemma 3, $\bar{g}$ and $\bar{a}$ have no common prime element. Suppose that $\mathfrak{b}=\mathfrak{O}p_1+\mathfrak{O}p_2$, and we shall prove that $\mathfrak{b}$ has no $\mathfrak{m}$-primary component. From $\varphi(\mathfrak{O}^*\mathfrak{b})=\bar{\mathfrak{O}}\bar{g}\cdot\bar{a}+\bar{\mathfrak{O}}\varphi(p_2)$, we have $\varphi(\mathfrak{O}^*\mathfrak{b})=\bar{\mathfrak{O}}\bar{g}\frown(\bar{\mathfrak{O}}\bar{a}+\bar{\mathfrak{O}}\varphi(p_2))$, since $\bar{\mathfrak{O}}\bar{a}:\bar{\mathfrak{O}}\bar{g}=\bar{\mathfrak{O}}\bar{a}$ and $\bar{\mathfrak{O}}\bar{g}\ni\varphi(p_2)$. By Lemma 3, $\bar{a}$ and $\varphi(p_2)$ have no common prime element, therefore $\bar{\mathfrak{O}}\bar{a}+\bar{\mathfrak{O}}\varphi(p_2)$ is unmixed and of rank 2 ($<$dim $\bar{\mathfrak{O}}$). Since ranks of components of $\varphi(\mathfrak{O}^*\mathfrak{b})$ are not greater than 2, ranks of components of $\mathfrak{O}^*\mathfrak{b}+\mathfrak{O}^*v^*$ are not greater than 3 ($<$dim $\mathfrak{O}^*$). This means $\mathfrak{O}^*\mathfrak{b}+\mathfrak{O}^*v^*$ has no $\mathfrak{m}^*$-primary component, hence $\mathfrak{c}=\mathfrak{O}\frown(\mathfrak{O}^*\mathfrak{b}+\mathfrak{O}^*v^*)$ has no $\mathfrak{m}$-primary components by Lemma 2. Since $\mathfrak{c}\supset\mathfrak{b}$, we have $\mathfrak{O}^*\mathfrak{b}+\mathfrak{O}^*v^*\supset\mathfrak{O}^*\mathfrak{c}\supset\mathfrak{O}^*\mathfrak{b}$, this implies $\mathfrak{O}^*\mathfrak{c}=\mathfrak{O}^*\mathfrak{b}$ because $\mathfrak{O}^*\mathfrak{c}$ has no $\mathfrak{m}^*$-components. Hence $\mathfrak{O}^*\mathfrak{b}(=\mathfrak{O}^*\mathfrak{c})$ has no $\mathfrak{m}^*$-component, consequently $\mathfrak{b}$ has no $\mathfrak{m}$-component, and therefore, $\mathfrak{O}^*\mathfrak{b}:\mathfrak{O}^*v^*=\mathfrak{O}^*\mathfrak{b}$.

Since $\bar{\mathfrak{O}}\varphi(p_1):\bar{\mathfrak{O}}\varphi(p_2)=\bar{\mathfrak{O}}\bar{a}$, we can find $\bar{b}$ which satisfies $\bar{b}\varphi(p_1)-\bar{a}\varphi(p_2)=0$. Let a^* and b^* be elements of $\mathfrak{O}^*$ such that $\varphi(a^*)=a$, $\varphi(b^*)=b$, then we have $b^*p_1-a^*p_2\in\mathfrak{O}^*v^*$, thus we have $b^*p_1-a^*p_2\in\mathfrak{O}^*\mathfrak{b}\cdot v^*$ since $\mathfrak{O}^*\mathfrak{b}:\mathfrak{O}^*v^*=\mathfrak{O}^*\mathfrak{b}$. Therefore we have $b^*p_1-a^*p_2=v^*(c^*p_1+d^*p_2)$, consequently we have $b_0^*p_1-a_0^*p_2=0$, where $b_0^*=b^*-v^*c^*$, $a_0^*=a^*+v^*d^*$. Hence $a_0^*\in\mathfrak{O}^*p_1:\mathfrak{O}^*p_2=\mathfrak{O}^*\mathfrak{a}$. On the other hand, from the equation $\varphi(a_0^*)=\varphi(a^*)=\bar{a}$, we have $\mathfrak{O}^*\mathfrak{a}\subset\varphi^{-1}(\bar{\mathfrak{O}}\bar{a})=\mathfrak{O}^*a_0^*+\mathfrak{O}^*v^*$, this implies that $\mathfrak{O}^*\mathfrak{a}=\mathfrak{O}^*a_0^*$. Since $\mathfrak{a}\not\subset\mathfrak{p}$, we have $a_0^*\notin\mathfrak{O}^*\mathfrak{p}$, and $p_1\in\mathfrak{O}^*\mathfrak{a}=\mathfrak{O}^*a_0^*$ implies that $p_1\in\mathfrak{O}^*\mathfrak{p}\cdot\mathfrak{O}^*a_0^*\subset\mathfrak{O}^*\mathfrak{p}\cdot\mathfrak{m}^*$, consequently $p_1\in\mathfrak{O}\frown\mathfrak{O}^*\mathfrak{p}\cdot\mathfrak{m}^*=\mathfrak{p}\cdot\mathfrak{m}$, thus we have obtained contradiction.

References

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