71. *Remarks on My Previous Paper on Congruence Zeta-Functions*

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1. First I want to give a correction of Lemma 2 in my previous paper [1].

Lemma. *Let H be a finite group of order h and χ be an irreducible character of H. Then we have*

$$\sum_{\tau\in H}\{\chi(\tau)^2-\chi(\tau^2)\}=0 \text{ or } 2h.$$

Moreover the second case occurs only if χ is real and the degree of χ is even.

Proof. Let $F:\tau\rightarrow F(\tau)=(a_{ij}(\tau))$ be an irreducible representation of H with the character χ. Then $F^*:\tau\rightarrow F^*(\tau)=(a^*_{ij}(\tau))={}^tF(\tau^{-1})=(a_{ji}(\tau^{-1}))$ is also an irreducible representation of H with the character $\bar{\chi}$. If F and F^* are not equivalent (i.e. χ is not real), the proof is the same as in [1]. Hence we may restrict ourselves to the case where F and F^* are equivalent; then we have $\sum_{\tau\in H}\chi(\tau)^2=h$. Let U be a non-singular matrix such that ${}^tF(\tau^{-1})=F^*(\tau)=U^{-1}F(\tau)U$ for all τ in H. Then we have $F(\tau)={}^tU^tF(\tau^{-1})^tU^{-1}={}^tUU^{-1}F(\tau)({}^tUU^{-1})^{-1}$ for all τ in H and so, by a lemma of Schur, ${}^tUU^{-1}=\rho E$, where E denotes the unit matrix. Considering the determinants of the both sides, we have $\rho^f=1$, where f is the degree of F. On the other hand, by ${}^tU=\rho U$, we have $U=\rho^2U$ and so $\rho^2=1$. Hence we have $\rho=\pm1$ and, especially, $\rho=1$ if f is odd. Let $U=(u_{ij})$ and $V=U^{-1}=(v_{ij})$. Then, as in [1], we have, by another lemma of Schur, $\sum_{\tau\in H}\chi(\tau^2)=\sum_{i,j,\tau}a_{ij}(\tau)a^*_{ij}(\tau^{-1})=\sum_{i,j,\tau}a_{ij}(\tau)\sum_{\mu,\nu}v_{i\mu}a_{\mu\nu}(\tau^{-1})u_{\nu j}=\sum_{i,j}\sum_{\mu,\nu}v_{i\mu}u_{\nu j}\sum_{\tau}a_{ij}(\tau)a_{\mu\nu}(\tau^{-1})=h/f\cdot\sum_{i,j}v_{ij}u_{ij}=h/f\cdot\mathrm{tr}(U^{-1t}U)=h/f\cdot\mathrm{tr}(\rho E)=\pm h$.

2. Let A/V be a Galois covering of degree n, defined over a finite field k with q elements, where A is an abelian variety and V is a normal, projective variety of dimension r; let G be the Galois group. Let Ξ be the character of the representation $M_l\,|\,G$ (the restriction of the l-adic representation of A to G) of G. Then, by the above lemma, $1/2n\cdot\sum_{\sigma\in G}\{\Xi(\sigma)^2-\Xi(\sigma^2)\}$ is a non-negative rational integer. By the orthogonality relation of group-characters and the results in [1], we have the following statement, which gives a correction and a supplement to the last part of Theorem 1 in [1].

Theorem. *Let the notations be as explained above. Then the zeta-function $Z(u,V)$ of V over k has $1/2n\cdot\sum_{\sigma\in G}\{\Xi(\sigma)^2-\Xi(\sigma^2)\}$ poles on the circle $|u|=q^{-(r-1)}$. Moreover, if there exist actually such poles,* *at least one of them is either* $u=q^{-(r-1)}$ *or* $u=-q^{-(r-1)}$.

Let $Z^{(2)}(u, V)$ be the zeta-function of V over k_2, the extension of k of degree 2, i.e. a finite field with q^2 elements. Then it is easily verified that the poles of $Z^{(2)}(u, V)$ on the circle $|u|=(q^2)^{-(r-1)}$ are equal to the squares of those of $Z(u, V)$ on the circle $|u|=q^{-(r-1)}$ respectively. Hence, if there exist such poles of $Z^{(2)}(u, V)$, at least one of them is $u=(q^2)^{-(r-1)}$.

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Reference

[1] M. Ishida: On zeta-functions and L-series of algebraic varieties. II, Proc. Japan Acad., **34**, 395-399 (1958).