69. A Note on Wedderburn Decompositions of Compact Rings

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Let R be a topological ring with (Perlis-Jacobson) radical N. We say that R admits a *Wedderburn decomposition* if R can be expressed as the form $R=S+N$ (direct sum of S and N as modules) for some closed semisimple subring S. The subring S is called a *Wedderburn factor* of R. Recently Jans has given a necessary and sufficient condition that a compact ring with open radical admits a Wedderburn decomposition [1, Theorem 1]. In this note, we shall extend Jans' theorem to general compact rings. The proof of our theorem is essentially the same as that of Jans'. As consequences of the theorem, however, Corollaries 1 and 2 in [1] are generalized in the natural form.

The following lemma is readily seen from the proof of Theorem 1 in [1] and Theorem E in [5].

Lemma. *Let R be a compact ring with $(p_1\cdots p_n)x=0$ for every x in R and for fixed distinct primes $p_1,\cdots,p_n$. Then $R=R_{p_1}\oplus\cdots\oplus R_{p_n}$, the ring direct sum of closed ideals $R_{p_i}=\{x\mid x\in R,\ p_ix=0\}$. Moreover, each R_{p_i} has a Wedderburn factor S_{p_i}, and hence R has a Wedderburn factor $S=S_{p_1}\oplus\cdots\oplus S_{p_n}$.*

For brevity we shall say that an element x of a topological ring has the *property* (γ) if for every nucleus (=neighborhood of 0) U of the ring there exist distinct primes $p_1,\cdots,p_k$ such that $(p_1\cdots p_k)x\in U$.

We can now give the following

Theorem. *Let R be a compact ring with the radical N and let e^* be the identity element of the residue class ring $R^*=R/N$. Then R admits a Wedderburn decomposition if and only if e^* can be raised to an element e_1 of R having the property (γ).*

Proof. The "only if" part is clear, since any compact semisimple ring is (algebraically and topologically) isomorphic to a complete ring direct sum (with Tychonoff topology) of finite simple rings [2, Theorem 16].

We are going to prove the converse way. Let e be the idempotent element in the closure of the positive powers of e_1 [3, Lemma 3]. It is clear that e is mapped onto e^* by the natural homomorphism $R\to R^*$. We shall show that the idempotent e has the property (γ). Suppose that U is an arbitrary compact nucleus in R. Let W be a nucleus such that $W\subset U$, $W\cdot R\subset U$. Since e_1 has the property (γ), there exist distinct primes $p_1,\cdots,p_n$ such that $(p_1\cdots p_n)e_1\in W$. Therefore $(p_1\cdots p_n)e_1^m \in W\cdot R \subset U$ for all positive integers m. It follows that $(p_1\cdots p_n)e \in U$, and e has the property (γ).

Now let $R=eRe+e\mathfrak{l}+\mathfrak{r}e+(\mathfrak{l}\frown\mathfrak{r})$ be the two-sided Peirce decomposition of R relative to the idempotent e, where $\mathfrak{l}$ and $\mathfrak{r}$ are the sets of left and right annihilators of e respectively, that is, $\mathfrak{l}=\{x \mid x\in R, xe=0\}$ and $\mathfrak{r}=\{y \mid y\in R, ey=0\}$. Since the latter three terms of the decomposition are contained in the radical N, it is sufficient to find a Wedderburn factor for eRe. Hence we may assume, without loss of generality, that the ring R has the identity which has the property (γ). In this case R has a system of ideal nuclei since R is totally disconnected [2, Theorem 8 and Lemma 9], and every element of R has the property (γ).

Let $\{M_\alpha, M_\beta,\cdots\}$ be the system of ideal nuclei in R. We denote by $R_\alpha, R_\beta,\cdots$ the residue class (finite) rings R/M_α, $R/M_\beta,\cdots$, respectively. Then R is the inverse limit of the system $\{R_\alpha, f_\beta^\alpha\}$ where each f_β^α is the natural homomorphism from R_α onto R_β, $M_\alpha\subset M_\beta$. Let $\{q_1=2, q_2=3, q_3,\cdots\}$ be the set of all primes such that $q_1<q_2<q_3<\cdots$. Since the identity of R has the property (γ) and M_α are ideals, for each $M_\alpha(\neq R)$ we can find a prime $q_{n(\alpha)}$ such that $(q_1q_2\cdots q_{n(\alpha)})x\in M_\alpha$ for every $x\in R$, but $(q_1q_2\cdots q_{n(\alpha)-1})z\notin M_\alpha$ for some $z\in R$. (Here it is assumed that $q_0=1$.) Then, by Lemma, the ring R_α is the ring direct sum of ideals $R_{\alpha i}$ where each $R_{\alpha i}$ consists of elements of R_α of additive order q_i $(i=1,2,\cdots,n(\alpha))$ if $R_{\alpha i}\neq 0$. It should be noticed that some $R_{\alpha j}$ $(1\leqq j<n(\alpha))$ may be 0. Now let us denote by $f_{\beta i}^\alpha$ the restrictions of f_β^α on $R_{\alpha i}$. Then it is not hard to see that the decompositions of R_α and the inverse system $\{R_\alpha, f_\beta^\alpha\}$ satisfy all the conditions of [4, Theorem 2]. Hence if we denote by R_i the inverse limit of the system $\{R_{\alpha i}, f_{\beta i}^\alpha\}$ $(i=1,2,\cdots)$, then R_i are closed ideals of R and R is the complete ring direct sum (with Tychonoff topology) of R_i: $R=\sum\oplus R_i$. It is clear that $q_ix=0$ for every x in R_i. Hence, by [5, Theorem 5E], R_i have a Wedderburn factor S_i. Then the complete direct sum $S=\sum\oplus S_i$ is the required Wedderburn factor in R.

From the proof of the theorem we can also obtain the following two corollaries:

Corollary 1. *If R is a compact ring, then R admits a Wedderburn decomposition if and only if the radical N has a complementary subgroup.*

Corollary 2. *If R is a compact ring with identity, then R admits a Wedderburn decomposition if and only if R is the complete ring direct sum (with Tychonoff topology) of algebras over finite fields.*

References

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