94. *A Tauberian Theorem for Fourier Series*

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1. Let $\varphi(t)$ be an even function, integrable in Lebesgue sense, periodic of period 2π, and let

$$\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt$$

and

$$s_n=\frac{1}{2}a_0+\sum_{\nu=1}^{n}a_\nu.$$

Hardy and Littlewood [1] have proved that if

$$\int_0^t|\varphi(u)|\,du=o\left(t\Big/\log\frac{1}{t}\right)\qquad(t\to 0),$$

and if for some positive δ

$$a_n>-An^{-\delta},\quad A>0,$$

then

$$s_n\to 0\quad\text{as}\quad n\to\infty.$$

In this paper we shall prove a converse

THEOREM. If $\sum a_n$ is summable to zero in Abel sense, and

$$\sum_{\nu=n}^{2n}|a_\nu|=o(1/\log n)\qquad(n\to\infty),\tag{1}$$

and if for some positive ρ,

$$\varphi'(t)>-At^{-\rho}\qquad(0<t<t_0),\tag{2}$$

where A is a positive constant independent of t, then

$$\varphi(t)\to 0\qquad(t\to 0).$$

2. Proof of the theorem. We require a

LEMMA. If $\sum u_n$ is summable in Abel sense, and if

$$u_{n+1}+u_{n+2}+\cdots+u_{n+\nu}>-K\quad(\nu=1,2,\cdots,n),$$

where K is a positive constant, then the series $\sum u_n$ converges to the same sum.

This is Lemma 2, slightly modified, of Szász [2].

For the proof of our Theorem, using the argument in Yano [3], we begin with the identities

$$\varphi(t)=\frac{1}{h}\int_0^h\varphi(t+u)du-\frac{1}{h}\int_0^h[\varphi(t+u)-\varphi(t)]du\tag{3}$$

and

$$\varphi(t)=\frac{1}{h}\int_0^h\varphi(t-u)\,du+\frac{1}{h}\int_0^h[\varphi(t)-\varphi(t-u)]du,\tag{4}$$

where $0<h<t$. The condition (2) implies

$$\varphi(t+u)-\varphi(t)=\int_0^u \varphi'(t+\mathrm{v})d\mathrm{v}>-Aut^{-\rho},$$

for $0<u<h$. Thus, letting

$$h=t^{\rho+1}, \tag{5}$$

we have, from (3)

$$\varphi(t)<\frac{1}{h}\int_0^h \varphi(t+u)du+At.$$

Similarly, from (4)

$$\varphi(t)>\frac{1}{h}\int_0^h \varphi(t-u)du-At.$$

Hence, if it is shown that

$$\lim_{t\to 0}\frac{1}{h}\int_0^h \varphi(t\pm u)du=0, \tag{6}$$

where h is defined by (5), then the result $\varphi(t)\to 0$ as $t\to 0$ follows immediately from the above two inequalities.

Now, replacing $\varphi(t+u)$ by its Fourier series,

$$\begin{aligned}\frac{1}{h}\int_0^h \varphi(t+u)\,du&=\frac{1}{h}\int_0^h\Big[\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}a_\nu\cos\nu(t+u)\Big]du\\ &=\frac{1}{h}\int_0^h\Big(\frac{1}{2}a_0+\sum_{\nu=1}^{n-1}\Big)du+\frac{1}{h}\int_0^h\Big(\sum_{\nu=n}^{N}\Big)du+\frac{1}{h}\int_0^h\Big(\sum_{\nu=N+1}^{\infty}\Big)du \qquad (7)\\ &=\mathrm{S}_1+\mathrm{S}_2+\mathrm{S}_3,\quad \text{say,}\end{aligned}$$

where we assume that, for fixed small $t>0$,

$$n=\Big[\frac{1}{t}\Big]\quad\text{and}\quad N=[n^{\rho+2}]. \tag{8}$$

Then, by Abel's transformation

$$\begin{aligned}\mathrm{S}_1&=\frac{1}{h}\int_0^h\Big[\frac{1}{2}a_0+\sum_{\nu=1}^{n-1}a_\nu\cos\nu(t+u)\Big]du\\ &=\sum_{\nu=0}^{n-1}\mathrm{s}_\nu\cdot\frac{1}{h}\int_0^h[\cos\nu(t+u)-\cos(\nu+1)(t+u)]du\\ &\qquad+\mathrm{s}_{n-1}\cdot\frac{1}{h}\int_0^h\cos n(t+u)du.\end{aligned}$$

And

$$\begin{aligned}&\frac{1}{h}\int_0^h[\cos\nu(t+u)-\cos(\nu+1)(t+u)]du\\ &\qquad=\frac{2}{h}\int_0^h\sin\Big(\nu+\frac{1}{2}\Big)(t+u)\sin\frac{1}{2}(t+u)du.\end{aligned}$$

Since the integrand in the last integral is positive and increasing with u in $(0,h)$ for $0<t+h<\pi/2(n+1)$ and $0\leqq\nu\leqq n$, the last expression is, by the second mean-value theorem,

$$\frac{2}{h}\sin\left(\nu+\frac{1}{2}\right)(t+h)\sin\frac{1}{2}(t+h)\int_{h_1}^{h}du \qquad (0<h_1<h)$$
$$=\theta_\nu\left(\nu+\frac{1}{2}\right)(t+h)^2<4\left(\nu+\frac{1}{2}\right)t^2 \qquad (0<\theta_\nu<1).$$

Hence

$$|S_1|<4\sum_{\nu=0}^{n-1}|s_\nu|\left(\nu+\frac{1}{2}\right)t^2+|s_{n-1}|.$$

On the other hand, the two conditions (1) and the Abel summability of $\sum a_n$ to the sum zero imply $\lim s_n=0$ by the above lemma. So, we may suppose that

$$|s_\nu|<K \ (\nu\geqq 0) \quad \text{and} \quad |s_\nu|<\varepsilon \ (\nu\geqq n_0).$$

For $n>n_0$, then,

$$|S_1|<2Kn_0^2t^2+2\varepsilon n^2t^2+\varepsilon<2Kn_0^2t^2+3\varepsilon,$$

by (8), and this is less than 4ε for $t<(\varepsilon/2K)^{1/2}/n_0$. Next

$$|S_2|=\left|\frac{1}{h}\sum_{\nu=n}^{N}a_\nu\int_0^h\cos\nu(t+u)du\right|<\sum_{\nu=n}^{N}|a_\nu|.$$

And, since by (8)

$$N=[n^{\rho+2}]\leqq n\cdot e^{(\rho+1)\log n}<n\cdot 2^{[2(\rho+1)\log n]},$$

and assuming that

$$(9) \qquad \sum_{\mu=\nu}^{2\nu}|a_\mu|<\varepsilon/\log\nu \qquad (\nu\geqq n_0>1),$$

which is permissible by (1), we have

$$|S_2|<\sum_{k=0}^{[2(\rho+1)\log n]-1}\ \sum_{\nu=2^kn}^{2^{k+1}n}|a_\nu|<\varepsilon\sum_{k=0}^{[2(\rho+1)\log n]-1}\frac{1}{\log(2^kn)}$$
$$\sim\varepsilon\int_0^{2(\rho+1)\log n}\frac{dx}{\log(2^xn)}<2\varepsilon\log(2\rho+3).$$

Further,

$$|S_3|=\left|\frac{1}{h}\sum_{\nu=N+1}^{\infty}a_\nu\int_0^h\cos\nu(t+u)du\right|$$
$$\leqq\frac{2}{h}\sum_{\nu=N+1}^{\infty}\frac{|a_\nu|}{\nu}\leqq\frac{2}{h}\sum_{k=0}^{\infty}\sum_{\nu=2^kN}^{2^{k+1}N}\frac{|a_\nu|}{\nu}$$
$$<\frac{2}{h}\sum_{k=0}^{\infty}\frac{1}{2^kN}\sum_{\nu=2^kN}^{2^{k+1}N}|a_\nu|,$$

which does not exceed, by (9),

$$\frac{2}{h}\sum_{k=0}^{\infty}\frac{1}{2^kN}\cdot\frac{\varepsilon}{\log(2^kN)}<\frac{4\varepsilon}{hN\log N}<\frac{4\varepsilon}{hn^{\rho+2}}\sim 4\varepsilon t,$$

since $n=[1/t]$ and $h=t^{\rho+1}$. Combining the above estimations of S's with (7), we have

$$\lim_{t\to 0}\frac{1}{h}\int_0^h\varphi(t+u)du=0.$$

Similarly,

$$\lim_{t\to 0}\frac{1}{h}\int_0^h \varphi(t-u)du=0,$$

and we get (6), which completes the proof.

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References

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