128. A Remark on a Theorem of J. P. Serre

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1. The purpose of this note is to prove the following

Theorem. *Let p be an odd prime, and let X be an arcwise- and simply-connected topological space satisfying*

i) $H_i(X, Z)$ *is finitely generated for all* $i>0$,

ii) $H_i(X, Z_p)=0$ *for all sufficiently large* i,

iii) $H_i(X, Z_p)\neq 0$ *for some* $i>0$.

Then there exist infinitely many values of i such that $\pi_i(X)$ has a subgroup isomorphic to Z or Z_p.

If we apply this theorem to $X=S^n$, a sphere of dimension $n\geq 2$, we obtain the result that for each S^n there exist infinitely many values of i such that the p-component of $\pi_i(S^n)$ is not zero and thus solve affirmatively Problem 12 of W. S. Massey.[1]

The above theorem was proved by J. P. Serre in the case $p=2$.[2] Our method of proof is a modification of that of Serre by using the results on $H_*(\pi, n; Z_p)$ due to H. Cartan.[3]

Throughout this note p is assumed to denote an odd prime.

2. **Lemma.** Let $n\geq 1$, and let π be a finitely generated abelian group. Then

i) $\vartheta(\pi, n; t)=\sum_{i=0}^{\infty}(\dim H_i(\pi, n; Z_p))t^i$ converges in the disk $|t|<1$.

ii) Setting

$$\varphi(\pi, n; x)=\log_p(\vartheta(\pi, n; 1-p^{-x})) \quad \text{for } 0\leq x<+\infty,$$

we have the following valuations. ($f(x)\sim g(x)$ means $\lim_{x\to+\infty} f(x)/g(x)=1$.)

$$\varphi(Z_{p^f}, n; x)\sim x^n/n!, \quad \varphi(Z, n; x)\sim\begin{cases} x^{n-1}/(n-1)! & \text{for } n\geq 2, \\ \log_p 2 & \text{for } n=1, \end{cases}$$

$$\varphi(Z_{q^f}, n; x)=0, \quad \text{where } q^f \text{ is a power of a prime } q(\neq p).$$

Proof of Lemma. We prove i) first. By the Künneth's relation $\vartheta(\pi+\pi', n; t)=\vartheta(\pi, n; t)\vartheta(\pi', n; t)$ for any finitely generated abelian groups π and π', it suffices to prove i) when $\pi=Z_{p^f}$ or Z or Z_{q^f}, where p^f and q^f mean the same as in ii). The case $\pi=Z_{q^f}$ is trivial, since $\vartheta(Z_{q^f}, n; t)=1$. The following expression (1) of $\vartheta(Z_{p^f}, n; t)$ is

1) W. S. Massey: Some problems in algebraic topology and the theory of fibre bundles, Ann. Math., **62**, 327–359 (1955).

According to this article, Problem 12 was also solved affirmatively by I. M. James.

2) J. P. Serre: Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, Comment. Math. Helv., **27**, 198–232, Theorem 10 (1953).

3) H. Cartan: Séminaire H. Cartan, E. N. S., 1954–1955.

easily set up.[4)]

$$(1)\qquad \vartheta(Z_{p^f}, n; t)=\prod_{h(\ )=n,\ h_1'+u_1\geq 1} 1/(1-t^{d\binom{2h_1'\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}) \prod_{h(\ )=n,\ h_1\ \mathrm{odd}} (1+t^{d\binom{h_1\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}).$$

The right hand side of (1) is to be regarded as the product of the two obvious formal power series

$$\vartheta_1(Z_{p^f}, n; t)=\prod_{h(\ )=n,\ h_1'+u_1\geq 1} 1/(1-t^{d\binom{2h_1'\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}) \quad \text{and}$$

$$\vartheta_2(Z_{p^f}, n; t)=\prod_{h(\ )=n,\ h_1\ \mathrm{odd}} (1+t^{d\binom{h_1\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}),$$

both power series being obtained by expanding formally the infinite product over all the indicated matrices, in which h_1' and h_i are non-negative integers for all $i\geq 1$, and $u_i=0$ or 1 for all $i\geq 1$. (Here the notations $d\begin{pmatrix} h_1 & 2h_2 & 2h_3\cdots \\ u_1 & u_2 & u_3\cdots \end{pmatrix}$ and $h\begin{pmatrix} h_1 & 2h_2 & 2h_3\cdots \\ u_1 & u_2 & u_3\cdots \end{pmatrix}$ denote the integers $h_1+2h_2p+2h_3p^2+\cdots+u_1\cdot 2+u_2\cdot 2p+u_3\cdot 2p^2+\cdots$ and $h_1+2h_2+2h_3+\cdots+u_1+u_2+u_3+\cdots$, respectively. The latter is abbreviated as $h(\ )$ when there is no confusion.) Since $\sum_{h(\ )=n} t^{d\binom{h_1\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}\leq\left(\sum_{i=1}^{\infty} t^i\right)^n$ for $0\leq t<1$, $\vartheta_1(Z_{p^f}, n; t)$ and $\vartheta_2(Z_{p^f}, n; t)$ converge in $0\leq t<1$. Therefore, $\vartheta(Z_{p^f}, n; t)$ converges, and (1) holds for $0\leq t<1$.

The corresponding expression of $\vartheta(Z, n; t)$ is obtained from (1) by excluding from the right hand side of (1) the factors corresponding to the matrices of the second kind. (A matrix $\begin{pmatrix} h_1 & 2h_2 & 2h_3\cdots \\ u_1 & u_2 & u_3\cdots \end{pmatrix}$ will be called to be "of the second kind", if $u_s=1$ for some s and $u_i=h_i=0$ for all $i>s$.) Therefore, $\vartheta(Z, n; t)$ converges in the disk $|t|<1$ by the above inequality, and the corresponding formula for $\vartheta(Z, n; t)$ holds for $0\leq t<1$.

We prove ii) now. We begin with the case $\pi=Z_{p^f}$. Setting

$$\vartheta^\circ(n; t)=\prod_{h(\ )=n} 1/(1-t^{d\binom{2h_1\ 2h_2\ 2h_3\cdots}{u_1\ u_2\ u_3\cdots}}) \quad \text{for } 0\leq t<1,$$

we have the following relations.

$$(2)\qquad \vartheta^\circ(n; t)=\vartheta_1(Z_{p^f}, n; t)\vartheta^\circ(n; t^p) \quad \text{for } n\geq 1.$$

$$(3)\qquad \begin{aligned}\vartheta^\circ(n-1; t^p)\vartheta^\circ(n-2; t)\geq\vartheta_1(Z_{p^f}, n; t)\geq\vartheta^\circ(n-1; t^{p^2})\\ \times\vartheta^\circ(n-2; t^p) \quad \text{for } n\geq 2,\end{aligned}$$

where $\vartheta^\circ(0; t)=1/(1-t^2)$ and $\vartheta^\circ(0; t^p)=1$. Setting further

$$\varphi_1(Z_{p^f}, n; x)=\log_p(\vartheta_1(Z_{p^f}, n; 1-p^{-x})) \quad \text{for } 0\leq x<+\infty,$$

$$\varphi^\circ(n; x)=\log_p(\vartheta^\circ(n; 1-p^{-x})) \quad \text{for } 0\leq x<+\infty,$$

we rewrite (2) and (3) as follows:

$$(2)'\qquad \begin{aligned}\varphi^\circ(n; x)=\varphi_1(Z_{p^f}, n; x)+\varphi^\circ\Big(n; x-1-\log_p\Big(1-p^{-1-x}\Big(\binom{p}{2}\\ -\binom{p}{3}p^{-x}+\cdots-p^{-(p-2)x}\Big)\Big)\Big),\end{aligned}$$

4) Cf. H. Cartan 3) Exposé 9.

$$\begin{aligned}&\varphi^\circ(n-1;x-1-\log_p(\ ))+\varphi^\circ(n-2;x)\geq\varphi_1(Z_{p^f},n;x)\\ (3)'\qquad &\geq\varphi^\circ\Big(n-1;x-2-\log_p\Big(1-\binom{p^2}{2}p^{-x-2}+\binom{p^2}{3}p^{-2x-2}-\cdots\\ &\qquad+p^{-(p^2-1)x-2}\Big)\Big)+\varphi^\circ(n-2;x-1-\log_p(\ )),\end{aligned}$$

where $\binom{m}{n}=m!/n!(m-n)!$ and $\log_p(\ )=\log_p\Big(1-p^{-1-x}\Big(\binom{p}{2}-\binom{p}{3}p^{-x}+\cdots-p^{-(p-2)x}\Big)\Big)$. In case $n=2$, $\varphi^\circ(0;x)$ and $\varphi^\circ(0;x-1-\log_p(\ ))$ in (3)′ are to be replaced by $x-\log_p(2-p^{-x})$ and 0, respectively. By an argument of Serre[5] it now follows from (2)′ that $\varphi_1(Z_{p^f},n;x)\sim x^n/n!$ implies $\varphi^\circ(n;x)\sim x^{n+1}/(n+1)!$ for $n\geq 1$. It is also clear from (3)′ that $\varphi^\circ(s;x)\sim x^{s+1}/(s+1)!$ for $s\leq n$ implies $\varphi_1(Z_{p^f},n+1;x)\sim x^{n+1}/(n+1)!$. Therefore, we obtain by induction on n that $\varphi_1(Z_{p^f},n;x)\sim x^n/n!$ for $n\geq 1$.

We now turn to $\vartheta_2(Z_{p^f},n;t)$. Setting

$$\vartheta'(n;t)=\prod_{h(\ )=n}\big(1+t^{d\left(\begin{smallmatrix}2h_1 & 2h_2\cdots\\ u_1 & u_2\cdots\end{smallmatrix}\right)}\big)\qquad\text{for } 0\leq t<1,$$

we have the following relations.

$$(4)\qquad \vartheta_2(Z_{p^f},n;t)\leq\vartheta'(n-1;t)\qquad\text{for } n\geq 2,$$

$$(5)\qquad \vartheta'(n;t)\leq\vartheta'(n-2;t)\vartheta'(n-1;t^p)\vartheta'(n;t^p)\qquad\text{for } n\geq 2,$$

where $\vartheta'(0;t)=1+t^2$.

Setting further

$$\varphi_2(Z_{p^f},n;x)=\log_p(\vartheta_2(Z_{p^f},n;1-p^{-x}))\qquad\text{for } 0\leq x<+\infty,$$
$$\varphi'(n;x)=\log_p(\vartheta'(n;1-p^{-x}))\qquad\text{for } 0\leq x<+\infty,$$

we rewrite (4) and (5) as follows:

$$(4)'\qquad \varphi_2(Z_{p^f},n;x)\leq\varphi'(n-1;x).$$

$$(5)'\qquad \begin{aligned}\varphi'(n;x)\leq\varphi'(n-2;x)&+\varphi'(n-1;x-1-\log_p(\ ))\\ &+\varphi'(n;x-1-\log_p(\ )),\end{aligned}$$

where $\log_p(\ )=\log_p\Big(1-p^{-1-x}\Big(\binom{p}{2}-\binom{p}{3}p^{-x}+\cdots-p^{-(p-2)x}\Big)\Big)$. By the above-mentioned argument of Serre it now follows from (5)′ that, given any $\varepsilon>0$,

$$\varphi'(n;x)\Big/\frac{x^n}{n!}\leq 1+\varepsilon\qquad\text{for all sufficiently large } x.$$

(The proof is by induction on n.) Together with (4)′ and the valuation $\varphi_1(Z_{p^f},n;x)\sim x^n/n!$ for $n\geq 1$, this completes the proof of ii), in case $\pi=Z_{p^f}$.

In case $\pi=Z$ $(n\geq 3)$, the proof is entirely analogous to the above and proceeds as follows. We first exclude from $\vartheta_1(Z_{p^f},n;t)$, $\vartheta_2(Z_{p^f},n;t)$, $\vartheta^\circ(n;t)$ and $\vartheta'(n;t)$ the factors corresponding to the matrices of the second kind, and we denote them by $\vartheta_1(Z,n;t)$, $\vartheta_2(Z,n;t)$, $\vartheta^\circ(Z,n;t)$ and $\vartheta'(Z,n;t)$, respectively. If, in each of the relations (2), (3), (4),

5) Cf. J. P. Serre 2), §3, 22°.

and (5), we replace $\vartheta_1(Z_{p^f}, n; t)$, etc. by $\vartheta_1(Z, n; t)$, etc., respectively, then the resulting relations still hold, and from these the desired conclusion follows by the same argument as in the case $\pi=Z_{p^f}$. For $n=1$ or 2 the proof is direct. The proof of the lemma is now complete.

3. If, in the original proof of Serre,[6)] we replace Z_2 by Z_p and use the above lemma instead of the corresponding one, then it applies to our theorem, and the theorem is established.