126. On Equivalence of Modular Function Spaces

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Let Ω be an abstract space and μ be a totally additive measure defined on a totally additive set class $\mathfrak{B}$ of subsets of Ω satisfying $\bigcup_{\mu(E)<\infty} E=\Omega$.

Let $\Phi(\xi, \omega)$ $(\xi\geqq 0,\ \omega\in\Omega)$ be a function satisfying the following conditions:

1) $0\leqq\Phi(\xi, \omega)\leqq\infty$ for all $\xi\geqq 0$, $\omega\in\Omega$;

2) $\Phi(\xi, \omega)$ is a measurable function on Ω for all $\xi\geqq 0$;

3) $\Phi(\xi, \omega)$ is a non-decreasing convex functions of $\xi\geqq 0$ for all $\omega\in\Omega$;

4) $\Phi(0, \omega)=0$ for all $\omega\in\Omega$;

5) $\Phi(\alpha-0, \omega)=\Phi(\alpha, \omega)$ for all $\omega\in\Omega$;

6) $\Phi(\xi, \omega)\to\infty$ as $\xi\to\infty$ for all $\omega\in\Omega$;

7) for any $\omega\in\Omega$, there exists $\alpha_\omega>0$ such that $\Phi(\alpha_\omega, \omega)<\infty$.

For any measurable function $x(\omega)$ $(\omega\in\Omega)$, $\Phi(|x(\omega)|, \omega)$ is also measurable. We shall denote by $L_\Phi(\Omega)$ the class of all measurable functions $x(\omega)$ $(\omega\in\Omega)$ such that, for some $\alpha=\alpha_x>0$,

$$\int_\Omega \Phi(\alpha|x(\omega)|, \omega)d\mu(\omega)<\infty.$$[1]

We write $x\geqq y$ $(x, y\in L_\Phi)$, if $x(\omega)\geqq y(\omega)$ for a.e.[2] on Ω, then L_Φ is a universally continuous semi-ordered linear space.

If we define a functional

$$m_\Phi(x)=\int_\Omega \Phi(|x(\omega)|, \omega)d\mu,$$

m_Φ satisfies all the modular conditions and furthermore m_Φ is monotone complete. Such a space L_Φ with m_Φ is said to be *a modular function space*.[3]

If $\overline{\Phi}(\eta, \omega)$ $(\eta\geqq 0,\ \omega\in\Omega)$ is, for every fixed $\omega\in\Omega$, the complementary function of Φ in the sense of H. W. Young, $\overline{\Phi}$ satisfies all the corresponding properties from 1) to 7) on Φ, and so, we have also a

1) For the integration, refer, for instance, H. Nakano [4].

2) Here "a.e. (almost everywhere)" means always that "except on some $A\in\mathfrak{B}$ which $\mu(E\cap A)=0$ for all $\mu(E)<\infty$".

3) Modulared function spaces were defined and discussed in H. Nakano [2, Appendices I, II]. For all other definitions and notations used in this note, see the same book, too.

modular function space $L_{\overline{\Phi}}$ with $m_{\overline{\Phi}}$, which is isometric to the conjugate modular space of L_Φ with m_Φ.

For two functions Φ and Ψ on Ω satisfying above the conditions, we say that L_Φ *is equivalent to* L_Ψ, if $L_\Phi = L_\Psi$.

In this note, we find a necessary and sufficient condition in order that a modular function space is equivalent to the other. Immediate consequence of the fact gives the condition in order that a modular function space is equivalent to an Orlicz space.

Lemma.[4] *Let R be an abstract modular space with two modulars m_1 and m_2, and m_1 be monotone complete.*

(I) *There exist $\varepsilon, \varepsilon', k, K, \gamma > 0$ such that*

(a) $m_2(kx) < \gamma$ *for all x with* $m_1(x) < \varepsilon$

(b) $m_2(kx) \leqq K m_1(x)$ *for all x with* $\varepsilon' \leqq m_1(x) < \varepsilon$.

(II) *If R is non-atomic, there exist $\varepsilon, k, K > 0$ such that*

$m_2(kx) \leqq K m_1(x)$ *for all x with* $m_1(x) \geqq \varepsilon$.

Proof. (I). (a) If (a) is not valid, there exists a sequence $0 \leqq x_\nu \in R$ $(\nu = 1, 2, \cdots)$ such that $m_1(x_\nu) \leqq \frac{1}{2^\nu}$, $m_2\left(\frac{1}{\nu} x_\nu\right) \geqq \nu$. Let $y_n = \bigcup_{\nu=1}^{n} x_\nu$ $(n = 1, 2, \cdots)$, then $0 \leqq y_n \uparrow_{n=1}^{\infty}$ and $m_1(y_n) \leqq 1$ $(n = 1, 2, \cdots)$. Therefore there exists $y_0 = \bigcup_{n=1}^{\infty} y_n$, because m_1 is monotone complete. On the other hand, for $\nu = 1, 2, \cdots$ $m_2\left(\frac{1}{\nu} y_0\right) \geqq m_2\left(\frac{1}{\nu} y_n\right) \geqq n$ for $n \geqq \nu$, which contradicts that m_2 is a modular.

(b) For any x with $\frac{\varepsilon}{2} \leqq m_1(x) < \varepsilon$, we have $m_2(kx) \leqq \gamma \leqq \frac{2\gamma}{\varepsilon} m_1(x)$ by (a).

(II) Let ε be the same as in (I). If $m_1(x) \geqq \varepsilon$, there exists an integer n such that $\varepsilon n \leqq m_1(x) < \varepsilon(n+1)$. We can decompose x into an orthogonal sequence x_ν $(\nu = 1, 2, \cdots, n+1)$ such that

$$x = \sum_{\nu=1}^{n+1} \oplus x_\nu \quad \text{and} \quad m_1(x_\nu) < \varepsilon,$$

because R is non-atomic. Therefore

$$m_2(kx) = \sum_{\nu=1}^{n+1} m_2(kx_\nu) < 2n\gamma \leqq \frac{2\gamma}{\varepsilon} m_1(x)$$

by (a) in (I).

First, we consider only the case that μ is non-atomic.

Theorem 1.[5] $L_\Phi(\Omega) \subseteqq L_\Psi(\Omega)$ *if and only if there exist $k, K > 0$ and $c(\omega) \in L_1(\Omega)$ such that*

$$(*) \qquad \Psi(k\xi, \omega) \leqq K\Phi(\xi, \omega) + c(\omega)$$

for all $\xi \geqq 0$ and a.e. on Ω.

4) The proof of this lemma has relations to results of [1] and [6].

5) This theorem is a generalization of Theorem 1a in [5, Chap. II, § 1].

Proof. It is clear that $(*)$ implies $L_\Phi \subseteqq L_\Psi$. We prove the converse. Let $0 \leqq \alpha_\nu$ $(\nu=1, 2, \cdots)$ be the system of all the positive rational numbers. For any $E \in \mathfrak{B}$ with $\mu(E) < \infty$ and for $\varepsilon, k, K > 0$ in (II), we put, for $\nu = 1, 2, \cdots$

$$(\sharp) \qquad E_\nu = \{\omega; \Psi(k\alpha_\nu, \omega) > K\Phi(\alpha_\nu, \omega)\} \frown E$$

and

$$x_\nu(\omega) = \alpha_\nu \chi_{E_\nu}(\omega)$$

respectively, where χ_{E_ν} is the characteristic function of E_ν.
We need to consider only on such ν that $\mu(E_\nu) \neq 0$ in the following.
Since $\Phi(\alpha_\nu, \omega) < \infty$ on E_ν by $(\sharp)$, we have

$$E_{\nu,n} = \{\omega; \Phi(\alpha_\nu, \omega) < n\} \frown E_\nu \uparrow_{n=1}^\infty E_\nu.$$

For all $n \geqq n_0$ where n_0 is sufficiently large such that $\mu(E_{\nu,n_0}) \neq 0$, we have $\alpha_\nu \chi_{E_{\nu,n}} \in L_\Phi$ and $m_\Phi(\alpha_\nu \chi_{E_{\nu,n}}) < \varepsilon$, if otherwise, the fact

$$m_\Psi(k\alpha_\nu \chi_{E_{\nu,n}}) = \int_{E_{\nu,n}} \Psi(k\alpha_\nu, \omega) d\mu$$
$$> K \int_{E_{\nu,n}} \Phi(\alpha_\nu, \omega) d\mu = K m_\Phi(\alpha_\nu \chi_{E_{\nu,n}})$$

contradicts (II).
Therefore, considering $\alpha_\nu \chi_{E_{\nu,n}} \uparrow_{n=1}^\infty$ and $m_\Phi(\alpha_\nu \chi_{E_{\nu,n}}) < \varepsilon$, we have $x_\nu = \bigcup_{n=1}^\infty \alpha_\nu \chi_{E_{\nu,n}} \in L_\Phi$ and $m(x_\nu) < \varepsilon$ likewise by $(\sharp)$ and (II). Here, putting $y_n = \bigcup_{\nu=1}^n x_\nu$, we have a sequence of step functions $0 \leqq y_n \uparrow_{n=1}^\infty$ where $y_n = \sum_{\mu=1}^n \alpha_{\nu_\mu} \chi_{E^{(\mu)}}$ for $\alpha_{\nu_1} < \alpha_{\nu_2} < \cdots < \alpha_{\nu_n}$ with $\nu_\mu = \mu$ $(\mu = 1, 2, \cdots, n)$ and for the system of disjoint sets $E^{(\mu)} = E_{\nu_\mu} - \left(\bigcup_{\rho=1}^{\mu-1} E_{\nu_\rho}\right) \frown E_{\nu_\mu}$.
Since, for all $n = 1, 2, \cdots$

$$m_\Psi(ky_n) = \int_\Omega \Psi\left(k \sum_{\rho=1}^n \alpha_{\nu_\rho} \chi_{E^{(\rho)}}(\omega), \omega\right) d\mu$$
$$= \sum_{\mu=1}^n \int_{E^{(\rho)}} \Psi(k\alpha_{\nu_\rho}, \omega) d\mu > \sum_{\rho=1}^n \int_{E^{(\rho)}} k\Phi(\alpha_{\nu_\rho}, \omega) d\mu$$
$$= K \int_\Omega \Phi\left(\sum_{\rho=1}^n \alpha_{\nu_\mu} \chi_{E^{(\rho)}}(\omega), \omega\right) d\mu = K m_\Phi(y_n),$$

we have $m_\Phi(y_n) < \varepsilon$ by (a) and (II).
Therefore $y_E = \bigcup_{n=1}^\infty y_n \in L_\Phi$ because m_Φ is monotone complete, and furthermore $m_\Phi(y_E) \leqq \varepsilon$. Namely $y_E \in L_\Psi$ by the hypothesis and $m_\Psi(ky_E) < \gamma$ by (a) in (I).
Now, we have, for all $n = 1, 2, \cdots$

$$\Psi(k\alpha_n, \omega) \leqq \begin{cases} \Psi(ky_E(\omega), \omega) & \text{for all } \omega \in E_n \\ K\Phi(\alpha_n, \omega) & \text{for all } \omega \in E - E_n \frown E. \end{cases}$$

The system $\{y_E\}$ in which every y_E is determined depending on $E \in \mathfrak{B}$

with $\mu(E)<\infty$ by the above-stated way constitutes a directed system $0\leqq y_E\uparrow_{\mu(E)<\infty}$, because for any two elements $y_E, y_F\in\{y_E\}$ we have $\mu(E\cup F)<\infty$ and $y_E\cup y_F=y_{E\cup F}$. Since $m(y_E)\leqq\varepsilon$ for any $\mu(E)<\infty$, and $E\uparrow_{\mu(E)<\infty}\Omega$ there exists $y_0=\bigcup_{\mu(E)<\infty} y_E\in L_\Phi$ with $m_\Phi(y_0)\leqq\varepsilon$ which is defined for all on Ω, and so, $y_0\in L_\Psi$ and $m_\Psi(ky_0)<\gamma$ by the same reason stated above.

Thus, we have for all positive real numbers $\xi\geqq 0$

$$\Psi(k\xi, \omega)\leqq K\Phi(\xi, \omega)+\Psi(ky_0(\omega), \omega) \quad \text{for a.e. on } \Omega$$

by 5). $\Psi(ky_0(\omega), \omega)$ is no other than $c(\omega)$ in $(*)$.

Corollary 1. *L_Φ is equivalent to L_Ψ if and only if there exist $k_1, k_2, K_1, K_2>0$ and $c\in L_1$ such that*

$$|K_1\Phi(k_1\xi, \omega)-K_2\Phi(k_2\xi, \omega)|\leqq c(\omega)$$

for all $\xi\geqq 0$ and a.e. on Ω.

Corollary 2. (1) *Let $L_{M(\xi)}(\Omega)$ be an Orlicz space defined on Ω by a function $M(\xi)$ $(\xi\geqq 0)$.*

$L_\Phi\subseteq L_M$ if and only if there exist $k, K>0$ and $c(\omega)\in L_1(\Omega)$ such that

$$\Phi(k\xi, \omega)\leqq KM(\xi)+c(\omega)$$

for all $\xi\geqq 0$ and a.e. on Ω.

(2) *Let $L_{p(\omega)}(\Omega)$ be a modular function space which is of unique spectra*[6] *defined on Ω by a measurable function $1\leqq p(\omega)\leqq\infty$ $(\omega\in\Omega)$.*

$L_\Phi\subseteqq L_{p(\omega)}$ if and only if there exist $k, K>0$ and $c\in L_1$ such that

$$\Phi(k\xi, \omega)\leqq K\xi^{p(\omega)}+c(\omega)$$

for all $\xi\geqq 0$ and a.e. on Ω.

Next, we consider the case that μ is atomic.

If μ is atomic, we can assume $\mu(\omega)=1$ for all $\omega\in\Omega$ without loss of generality. And $m_\Phi(x)=\sum_{\omega\in\Omega}\Phi(|x(\omega)|, \omega)$ for all $x\in L_\Phi$.

Theorem 2. *$L_\Phi\subseteqq L_\Psi$ if and only if there exist $k, K>0$, $c(\omega)\geqq 0$ $(\omega\in\Omega)$ with $\sum_{\omega\in\Omega}c(\omega)<\infty$ and ξ_ω $(\omega\in\Omega)$ which is a system of numbers satisfying, for any $x\in L_\Phi$ with $m_\Phi(x)<\infty$, $|x(\omega)|\leqq\xi_\omega$ except on some finite subset of Ω, such that*

$$(**) \qquad \Psi(k\xi, \omega)\leqq K\Phi(\xi, \omega)+c(\omega)$$

for all $\xi\leqq\xi_\omega$ and $\omega\in\Omega$.

Proof. If $(**)$ is valid, for any $x\in L_\Phi$, since there exists $\alpha>0$

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6) These spaces were defined and discussed precisely in H. Nakano [3, §89], that is,

$$L_{p(\omega)}(\Omega)=\left\{x\colon \int_\Omega |\alpha x(\omega)|^{p(\omega)}d\mu<\infty \quad \text{for some } \alpha=\alpha_x>0\right\}$$

with the modular

$$m(x)=\int_\Omega \frac{1}{p(\omega)}|x(\omega)|^{p(\omega)}d\mu$$

for a measurable function $1\leqq p(\omega)\leqq\infty$ on Ω.

with $m_\Phi(\alpha x)<\infty$, we can find a finite subset $F\subset\Omega$ such that $\alpha|x(\omega)|\leqq\xi_\omega$ for all $\omega\bar{\in}F$ and

$$\Psi(k\alpha\,|\,x(\omega)\,|,\omega)\leqq K\Phi(\alpha\,|\,x(\omega)\,|,\omega)+c(\omega)$$

for all $\omega\bar{\in}F$.
Furthermore, there exists k' with $0<k'\leqq\alpha k$ such that $\Psi(k'|x(\omega)|,\omega)<\infty$ for all $\omega\in F$, because $0\leqq|x(\omega)|<\infty$ and 7) on Ψ.
Thus, we have

$$\begin{aligned}\sum_{\omega\in\Omega}&\Psi(k'\,|\,x(\omega)\,|,\omega)\\ &\leqq\sum_{\alpha|x(\omega)|\leqq\xi_\omega}K\Phi(\alpha\,|\,x(\omega)\,|,\omega)+\sum_{\alpha|x(\omega)|\leqq\xi_\omega}c(\omega)+\sum_{\alpha|x(\omega)|>\xi_\omega}\Psi(k'\,|\,x(\omega)\,|,\omega)\\ &\leqq K\sum_{\omega\in\Omega}\Phi(\alpha\,|\,x(\omega)\,|,\omega)+\sum_{\omega\in\Omega}c(\omega)+\sum_{\omega\in F}\Psi(k'\,|\,x(\omega)\,|,\omega)<\infty.\end{aligned}$$

Namely $x\in L_\Phi$.

Conversely, if $L_\Phi\subseteq L_\Psi$, then (**) is proved also by the analogous way to the case which μ is non-atomic as the following.

Let $0\leqq\alpha_\nu$ $(\nu=1,2,\cdots)$ be the system of all the positive rational numbers and let ε' and ε in (I) be $2\varepsilon'<\varepsilon$. We put, for all $\omega\in\Omega$,

$$\alpha_\omega=\sup\{\alpha;\alpha>0,\ \Psi(k\alpha,\omega)>K\Phi(\alpha,\omega),\ \Phi(\alpha,\omega)<\varepsilon'\}$$

and

$$x_0(\omega)=\bigcup_{\omega'\in\Omega}\alpha_{\omega'}\chi_{\omega'}(\omega)=\bigcup_{F\subset\Omega}\ \bigcup_{\omega'\in F}\alpha_{\omega'}\chi_{\omega'}(\omega)$$

where every $\chi_{\omega'}$ is the characteristic function of $\{\omega'\}$ and F is a finite subset of Ω, respectively.
Then, because, for any two elements $\omega_1,\omega_2\in\Omega$,

$$m_\Phi(\alpha_{\omega_1}\chi_{\omega_1}\cup\alpha_{\omega_2}\chi_{\omega_2})=\Phi(\alpha_{\omega_1},\omega_1)+\Phi(\alpha_{\omega_2},\omega_2)<2\varepsilon'<\varepsilon$$

and

$$m_\Psi(k\alpha_{\omega_1}\chi_{\omega_1}\cup\alpha_{\omega_2}\chi_{\omega_2})>Km_\Phi(\alpha_{\omega_1}\chi_{\omega_1}\cup\alpha_{\omega_2}\chi_{\omega_2})$$

we have $m_\Phi(\bigcup_{\omega\in F}\alpha_\omega\chi_\omega)<\varepsilon'$ by (I).
Since $\bigcup_{\omega\in F}\alpha_\omega\chi_\omega\uparrow_{F\subset\Omega}x_0$ and m_Φ is monotone complete, we have $x_0\in L_\Phi$ and $m_\Phi(x_0)\leqq\varepsilon'$. Namely $x_0\in L_\Psi$ by the hypothesis and $m_\Psi(kx_0)\leqq\gamma$ by (a) in (I).
On the other hand, putting

$$\xi_\omega=\sup_{\Phi(\xi,\omega)\leqq\varepsilon}\xi,$$

then for any $x\in L_\Phi$ with $m_\Phi(x)=\sum_{\omega\in\Omega}\Phi(|\,x(\omega)\,|,\omega)<\infty$ we can find a finite subset $F_x\in\Omega$ such that $\Phi(|\,x(\omega)\,|,\omega)\leqq\varepsilon$ for all $\omega\bar{\in}F_x$. Namely $|\,x(\omega)\,|\leqq\xi_\omega$ for all $\omega\bar{\in}F_x$.

Thus, we have, for all $\omega\in\Omega$

$$\Phi(k\xi,\omega)\leqq\begin{cases}\Psi(kx_0(\omega),\omega) & \text{for all } 0\leqq\xi<\alpha_\omega\\ K\Phi(\xi,\omega) & \text{for all } \alpha_\omega\leqq\xi\leqq\xi_\omega\end{cases}$$

by (b) in (I). Therefore (**) is proved by putting

$$c(\omega)=\Psi(kx_0(\omega),\omega).$$

Similar conditions to the corollaries to Theorem 1 on the case that

μ is non-atomic hold also in this case.

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