122. *On Finite Dimensional Quasi-norm Spaces*

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1959)

In this Note, we shall consider a *finite dimensional quasi-norm space* E*) *of order* r. Suppose that the dimension of E is n and let $e_1, e_2, \cdots, e_n$ be the bases of E. Then any element x of E may be written in the form

$$x=\lambda_1 e_1+\lambda_2 e_2+\cdots+\lambda_n e_n.$$

Let $\{x_m\}$ be a sequence of E, and let

$$x_m=\sum_{i=1}^{n}\lambda_i^m e_i.$$

If $\lambda_i^m\to\lambda_i$ $(m\to\infty)$ for every i,

$$\|x_m-x\|=\left\|\sum_{i=1}^{n}(\lambda_i^m-\lambda_i)e_i\right\|\leq\sum_{i=1}^{n}\|(\lambda_i^m-\lambda_i)e_i\|$$

$$\leq|\lambda_i^m-\lambda_i|^r\sum_{i=1}^{n}\|e_i\|\to 0 \qquad (m\to\infty).$$

Hence we have $x_m\to x$ $(m\to\infty)$.

Now we shall prove the following

Lemma. For any element $x=\sum_{i=1}^{n}\lambda_i e_i$ *of* E, *there is a positive number* H *such that*

$$|\lambda_i|^r\leq H\|x\|,$$

where H *depends on the base* e_i *of* E.

Proof. Let S be the unit sphere of n-dimensional space R^n. For $\Xi=(\lambda_1,\cdots,\lambda_n)$ we put $x(\Xi)=\sum_{i=1}^{n}\lambda_i e_i$, the linear independence of e_i and $\sum_{i=1}^{n}\lambda_i^2=1$ imply $x(\Xi)\neq 0$. As mentioned above, $\Xi^m\to\Xi$ $(m\to\infty)$ in R^n implies $x(\Xi^m)\to x(\Xi)$. Hence $x(\Xi)$ is continuous on the compact set S. Therefore we have $m=\operatorname{Min}_{\Xi\in S}\|x(\Xi)\|>0$.

Let $H=\dfrac{1}{m}$, and take a non-zero element $x=\sum_{i=1}^{n}x_i e_i$ of E

$$x'=\frac{1}{\sqrt{\sum_{i=1}^{n}\lambda_i^2}}x=\sum_{i=1}^{n}\mu_i x_i,$$

where

$$\mu_K=\frac{\lambda_K}{\sqrt{\sum_{i=1}^{n}\lambda_i^2}}.$$

From $\sum_{i=1}^{n}\mu_i^2=1$, we have $\|x'\|\geq m$. Hence

*) For details, see T. Konda [1], M. Pavel [2], and S. Rolewicz [3].

$$\|x\|=\left\|\sqrt{\sum_{i=1}^{n}\lambda_i^2}\,x'\right\|=\left(\sum_{i=1}^{n}\lambda_i^2\right)^{\frac{r}{2}}\|x'\|$$

$$\geq m\left(\sum_{i=1}^{n}\lambda_i^2\right)^{\frac{r}{2}}$$

and we have

$$|\lambda_i|^r\leq\frac{1}{m}\|x\|=H\|x\|.$$

This completes the proof of Lemma.

Let $x^m=\sum_{i=1}^{n}\lambda_i^m e_i$ be a sequence of E, and suppose that x^m converges to an element $x=\sum_{i=1}^{n}\lambda_i e_i$ in the sense of norm. Then

$$x^m-x=\sum_{i=1}^{n}(\lambda_i^m-\lambda_i)e_i.$$

By Lemma, we have, for $i=1,2,\cdots,n$,

$$|\lambda_i^m-\lambda_i|^r\leq H\|x^m-x\|.$$

Hence $\lambda_i^m\to\lambda_i$ $(m\to\infty)$ for every i.

If $\{x_m\}$ is a fundamental sequence, by Lemma,

$$|\lambda_i^p-\lambda_i^q|^r\leq H\|x_p-x_q\|\to 0 \qquad (p,q\to\infty)$$

and $\{\lambda_i^m\}$ $(i=1,2,\cdots,n)$ is a fundamental sequence. Hence the sequence $\{x_m\}$ converges to an element of E. This shows that E is complete. Therefore we have

Theorem. Any finite dimensional quasi-normed space is isomorphic to the Euclidean space.

References

[1] T. Konda: On quasi-normed space. I, Proc. Japan Acad., **35**, 340–342 (1959).
[2] M. Pavel: On quasi-normed spaces, Bull. Acad. Polon. Sci., cl. III, **5**, 479–484 (1957).
[3] S. Rolewicz: On a certain class of linear metric spaces, Bull. Acad. Polon. Sci., cl. III, **5**, 471–473 (1957).