4. On Predicates Expressible in the 1-Function Quantifier Forms in Kleene Hierarchy with Free Variables of Type 2*)

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The aim of this note is to show that there are certain non analogical properties between the 1-function quantified predicates (or the sets definable by such predicates) in Kleene hierarchy with free variables of types ≤ 1 and those with free variables of type 2.

In particular, there is no second separation (and hence, reduction) result for the classes of sets definable by the 1-function quantified predicates in Kleene hierarchy with free variables of type 2, in contrast to the case with free variables of types ≤ 1[1] in which the reduction theorem holds for the universal quantifier first side — namely, using the notation of Addison's [1], for Π_1^1 sets (and hence, as to separation, the first and the second separation theorems hold for the existential quantifier first side — namely, for Σ_1^1 sets).[2]

Throughout this paper, the ideas and the techniques of Kleene [8, in particular, 8.2–8.8] are used.

1. We assume familiarity with [5–8], and use the notation of them unless further references. Using the notation of [1] (in addition, with superscripts $m=0, 1, 2, \cdots$ exhibiting the maximal type of free variables of the predicates), we denote by $\Sigma_k^{r,m}$, $\Pi_k^{r,m}$ the classes of predicates (or sets) in Kleene hierarchies, according to the kind of the outermost quantifier, the highest type "r" of variables quantified, the number "k" of alternations of the quantifiers of type r and the highest type "m" of quantifier-free variables in the forms of Kleene's tables (cf. [6, (a) with free variables of types ≥ 0, p. 315] and [8, (c_1) p. 41]), respectively. For example, a predicate $P(a, \alpha, \mathbf{F})$ expressible in the form

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*) The contents with detailed proofs will appear in a forthcoming paper. As to Kleene hierarchies, it is referred to Kleene's excellent series of papers [4, 6–8] on hierarchies obtained by quantifying variables of recursive predicates.

1) Cf. [1, §5, Case 2].

2) Recently Addison [2, p. 352] has conjectured that on hierarchies based on quantification of higher type, under the assumption of the axiom of constructibility, the separation results are uniform according to the kind of the outermost quantifier for all types r ($r\geq 0$) and levels k ($k\geq 1$) except for the lone case when $r=1$, $k=1$. In fact, for Kleene hierarchies based on quantification of finite type, we can assure that Addison's conjecture holds, and see that the separation results are uniform also in types m ($m\leq r+1$) of quantifier-free variables under the same exception. As to the case when $r=1$, $k=1$, the author's result shows that the separation principle does not uniformly behave to types of quantifier-free variables.

$$(E\beta)(x)R(a, \alpha, \mathbf{F}, \beta, x)$$

with a general recursive R is of $\Sigma_1^{1,2}$.

2. As is well known, for $m \leq 1$, every predicate $P(\mathfrak{a})$[3] of $\Sigma_1^{1,m}(\Pi_1^{1,m})$ is expressible in the form $(E\beta)(x)R(\mathfrak{a}, \bar{\beta}(x))((\beta)(Ex)R(\mathfrak{a}, \bar{\beta}(x)))$ with a primitive recursive $R(\mathfrak{a}, u)$ (cf. for example [8, footnote 14)] or [7, XVII]):

$$(2.1) \qquad P(\mathfrak{a}) \equiv (E\beta)(x)R(\mathfrak{a}, \bar{\beta}(x))((\beta)(Ex)R(\mathfrak{a}, \bar{\beta}(x))).$$

Unlike the case for $m \leq 1$, there is in general no primitive recursive $R(\mathfrak{a}, u)$ such that (2.1) holds for a predicate $P(\mathfrak{a})$ of $\Sigma_1^{1,2}$ (and dually $\Pi_1^{1,2}$). Now, let $\Sigma_1^{1,2*}(\Pi_1^{1,2*})$ be the class of predicates $P(\mathfrak{a})$ of $\Sigma_1^{1,2}(\Pi_1^{1,2})$ for which (2.1) holds with a primitive recursive R. Then we have

Theorem 1. $\Sigma_1^{1,2*} \subsetneqq \Sigma_1^{1,2}(\Pi_1^{1,2*} \subsetneqq \Pi_1^{1,2})$.

In fact, we see that if $\lambda\mathbf{F}(E\beta)[\mathbf{F}(\beta)=0]$ (which is of $\Sigma_1^{1,2}$) were a predicate of $\Sigma_1^{1,2*}$, then it would be contradictory to the following

Lemma 1.[4] *Let* $\mathbf{F}_0$ *be a given type*-2 *object* (namely, a completely defined functional with one function variable α as argument) *which remains with a fixed value when the value of argument* α *is primitive recursive. Then any function* $\phi(a_1, \cdots, a_n)$ *primitive recursive in* $\mathbf{F}_0$ *is primitive recursive.*

The proof of this lemma is given by induction on the length of a primitive recursive description of $\phi(\mathfrak{a}^0, \mathbf{F})$.

3. Let $\mathbf{E}_1$ be the particular type-2 object defined thus:

$$\mathbf{E}_1(\alpha) = \begin{cases} 0 & \text{if } (\beta)(Ex)[\alpha(\bar{\beta}(x))=0], \\ 1 & \text{otherwise.} \end{cases}$$

Employing this object $\mathbf{E}_1$, we have the following theorems.

Theorem 2. *If* $\phi(\mathbf{F}, a_1, \cdots, a_n, \alpha)$ *is primitive recursive with a description of* $\mathbf{F}$*-height* h (cf. [8, p. 45]), *then* $\phi(\mathbf{E}_1, a_1, \cdots, a_n, \alpha)$ *is primitive recursive in* $\mathfrak{L}_h^\alpha$ *uniformly in* α.[5]

The proof of this theorem is obtained by the methods similar to the proof of [8, XLVI].

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3) $\mathfrak{a}^m$ denotes a list of variables of types $\leq m$. The superscript m is omitted if arbitrary or clear from context.

4) We have also a strengthened form of this lemma and of the fact that if $\phi(a_1, \cdots, a_n)$ is general (primitive) recursive in $\mathbf{F}_0$ (cf. [8, p. 48]) which is a given general (primitive) recursive object of type 2, then $\phi(a_1, \cdots, a_n)$ is general (primitive) recursive:

There is a primitive recursive function $\zeta(z, w)$ *with the following property. Let* z, w *be indices of* $\phi(\mathfrak{a}^0, \mathbf{F})$, $\theta(\alpha)$, *and* θ *be defined for all general recursive functions* α. *Then, for values of* $\mathfrak{a}^0, \mathbf{F}$ *such that* $\phi(\mathfrak{a}^0, \mathbf{F})$ *is defined and*

$(*)$ $\qquad \mathbf{F}(\alpha) = \theta(\alpha)$ *for any general recursive* α,

we have $\phi(\mathfrak{a}^0, \mathbf{F}) = \{\zeta(z, w)\}(\mathfrak{a}^0)$.

Thus: *Given a particular type*-2 *object* $\mathbf{F}_0$ *for which there is a general (primitive) recursive function* $\theta(\alpha)$ *such that* $(*)$ *holds, then any function* $\phi(a_1, \cdots, a_n)$ *general (primitive) recursive in* $\mathbf{F}_0$ *is general (primitive) recursive.*

5) For $\mathfrak{L}_h^\alpha$, cf. [7, p. 211] taking $Q(a) \equiv \alpha((a)_0) = (a)_1$.

Theorem 3. *When $m \leq 1$, any predicate $P(\mathfrak{a}^m)$ primitive recursive in $\mathbf{E}_1$ is of $\Sigma_2^{1,m} \cap \Pi_2^{1,m}$.*

Suppose $P(\mathfrak{a}^m)$ is primitive recursive in $\mathbf{E}_1$. Then the representing function $\phi(\mathfrak{a}^m)$ of $P(\mathfrak{a}^m)$ is primitive recursive in $\mathbf{E}_1$, and hence there is a primitive recursive function $\phi(\mathbf{F}, \mathfrak{a}^m)$ such that $\phi(\mathbf{E}_1, \mathfrak{a}^m) = \phi(\mathfrak{a}^m)$. By induction on the length of a description of ϕ, it can be shown that the predicate $\phi(\mathbf{E}_1, \mathfrak{a}^m) = w$ is of $\Sigma_2^{1,m} \cap \Pi_2^{1,m}$. For case 8, the techniques of [3] are used.

Next, let be given $P(\mathfrak{a})$ general recursive in $\mathbf{E}_1$. Then there is a partial recursive $\phi(\mathbf{F}, \mathfrak{a})$, say with index z, such that $\phi(\mathbf{E}_1, \mathfrak{a})$ is completely defined and is the representing function of $P(\mathfrak{a})$. Using [8, XXVI with (13)] for the predicate "$\phi(\mathbf{E}_1, \mathfrak{a}) = 0$", we have a corollary.

Corollary. *When $m \leq 1$, any predicate $P(\mathfrak{a}^m)$ general recursive in $\mathbf{E}_1$ is of $\Sigma_2^{1,m} \cap \Pi_2^{1,m}$.*

We can see that for any $y \in O_{2_O}$ the representing function of $\mathfrak{H}_y(a)$[6] is general recursive in $\mathbf{E}_1$. It is under investigation what subclass of $\Sigma_2^{1,m} \cap \Pi_2^{1,m}$ constitute the predicates $P(\mathfrak{a}^m)$ general recursive in $\mathbf{E}_1$ for $m \leq 1$.

4. Now, for a given type-2 object $\mathbf{F}_0$ we denote by $\Sigma_1^{1,m}[\mathbf{F}_0]$ ($\Pi_1^{1,m}[\mathbf{F}_0]$) the class of predicates $P(\mathfrak{a}^m)$ for which there is a predicate $R(\mathfrak{a}^m, \alpha, x)$ general recursive in $\mathbf{F}_0$ such that

$$P(\mathfrak{a}^m) \equiv (E\beta)(x)R(\mathfrak{a}^m, \beta, x)((\beta)(Ex)R(\mathfrak{a}^m, \beta, x)).$$

It holds an interesting theorem as follows.

Theorem 4. *For $m \leq 1$, $\Sigma_1^{1,m} = \Sigma_1^{1,m}[\mathbf{E}]$ ($\Pi_1^{1,m} = \Pi_1^{1,m}[\mathbf{E}]$),*[7] *$\Sigma_2^{1,m} = \Sigma_1^{1,m}[\mathbf{E}_1]$ ($\Pi_2^{1,m} = \Pi_1^{1,m}[\mathbf{E}_1]$).*

Let $m \leq 1$. Evidently, we have $\Sigma_1^{1,m} \subset \Sigma_1^{1,m}[\mathbf{E}]$. In the proof of the converse implication $\Sigma_1^{1,m}[\mathbf{E}] \subset \Sigma_1^{1,m}$, it is used that *for $m \leq 1$ any predicate $P(\mathfrak{a}^m)$ general recursive in $\mathbf{E}$ is of $\Sigma_1^{1,m} \cap \Pi_1^{1,m}$.* The latter is easily assured by the same methods as the first half of the proof for [8, XLVIII].

Next, to prove $\Sigma_2^{1,m} \subset \Sigma_1^{1,m}[\mathbf{E}_1]$ it is sufficient to show that, for example, the predicate $(E\beta)(\gamma)(Ex)T_1^{\alpha,\beta,1}(\bar{\gamma}(x), a, a)$ is of $\Sigma_1^{1,1}[\mathbf{E}_1]$.

The function

$$\tau(u, \alpha, \beta, a) = \begin{cases} 0 & \text{if } T_1^{\alpha,\beta,1}(u, a, a), \\ 1 & \text{otherwise} \end{cases}$$

is primitive recursive, and so is

$$\tau(\mathbf{F}, \alpha, \beta, a) = \mathbf{F}(\lambda u \tau(u, \alpha, \beta, a)).$$

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6) The notation for ordinals "O_{2_O}" and the hierarchy "$\mathfrak{H}_y$" have been introduced and discussed by Addison and Kleene in [3]. For these definitions, cf. [3, p. 1003].

7) $\mathbf{E}$ is the particular type-2 object defined by Kleene (cf. [8, p. 45]) thus:

$$\mathbf{E}(\alpha) = \begin{cases} 0 & \text{if } (Ex)[\alpha(x) = 0], \\ 1 & \text{otherwise.} \end{cases}$$

On giving $\mathbf{F}$ the fixed value $\mathbf{E}_1$, we have

(4.1) $(E\beta)[\tau(\mathbf{E}_1, \alpha, \beta, a)=0] \equiv (E\beta)(\gamma)(Ex)T_1^{\alpha,\beta,1}(\bar{\gamma}(x), a, a)$,

where $(E\beta)[\tau(\mathbf{E}_1, \alpha, \beta, a)=0]$ is of $\Sigma_1^{1,1}[\mathbf{E}_1]$ by the definition. The converse implication $\Sigma_1^{1,m}[\mathbf{E}_1] \subset \Sigma_2^{1,m}$ is proved by the corollary of Theorem 3.

5. In addition to [2, p. 352], we are on the point of showing a further example that $(\Sigma_1^{1,m}, \Pi_1^{1,m})$-predicates stand alone as peculiar.

Theorem 5. *It holds neither* $\mathrm{Sep}_{\mathrm{II}}(\Sigma_1^{1,2})$ *nor* $\mathrm{Sep}_{\mathrm{II}}(\Pi_1^{1,2})$.[8]

In fact, by $\overline{\mathrm{Sep}_{\mathrm{II}}}(\Sigma_2^{1,1})$[9] and $\Sigma_2^{1,1}=\Sigma_1^{1,1}[\mathbf{E}_1]$, we have $\overline{\mathrm{Sep}_{\mathrm{II}}}(\Sigma_1^{1,2})$. On the other hand, by $\overline{\mathrm{Sep}_{\mathrm{II}}}(\Pi_1^{1,1})$[9] and $\Pi_1^{1,1}=\Pi_1^{1,1}[\mathbf{E}]$, we have $\overline{\mathrm{Sep}_{\mathrm{II}}}(\Pi_1^{1,2})$.

By the above theorem, we have $\overline{\mathrm{Red}}(\Pi_1^{1,2})$ and $\overline{\mathrm{Red}}(\Sigma_1^{1,2})$ (cf. [9] or [1]). But it is open how behaves the first separation principle for $(\Sigma_1^{1,2}, \Pi_1^{1,2})$-sets.

Remark. Using [2, Theorem 1], we can prove under the assumption of the axiom of constructibility $\mathrm{Red}(\Sigma_k^{1,2})$ for $k>1$.

6. Let $\Sigma_1^{1,2\dagger}(\Pi_1^{1,2\dagger})$ be the class of predicates $P(\mathfrak{a})$ of $\Sigma_1^{1,2}(\Pi_1^{1,2})$ for which (2.1) holds with a general recursive R. By the corollary of Theorem 3, (4.1) and using the techniques of [3], we obtain the next theorem, in contrast with the case of $m \leq 1$.

Theorem 6. $\Sigma_1^{1,2\dagger} \subsetneqq \Sigma_1^{1,2}(\Pi_1^{1,2\dagger} \subsetneqq \Pi_1^{1,2})$.

In addition to Kleene [8, XLIX, (b)], we have by Theorem 2 and the estimation of hyperdegree[10] that *a general recursive predicate* $P(\mathfrak{a}^2)$ *is not in general expressible in the form* $(E\beta)(x)R(\mathfrak{a}^2, \bar{\beta}(x))$ (or $(\beta)(Ex)R(\mathfrak{a}^2, \bar{\beta}(x))$) *with a primitive recursive* $R(\mathfrak{a}^2, u)$. Hence we have

Theorem 7. $\Sigma_1^{1,2*} \subsetneqq \Sigma_1^{1,2\dagger}(\Pi_1^{1,2*} \subsetneqq \Pi_1^{1,2\dagger})$.

Unifing the above two theorems, we have $\Sigma_1^{1,2*} \subsetneqq \Sigma_1^{1,2\dagger} \subsetneqq \Sigma_1^{1,2}(\Pi_1^{1,2*} \subsetneqq \Pi_1^{1,2\dagger} \subsetneqq \Pi_1^{1,2})$ which gives also another proof of Theorem 1.

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8) For $\mathrm{Sep}_{\mathrm{I}}(Q)$, $\mathrm{Sep}_{\mathrm{II}}(Q)$ and $\mathrm{Red}(Q)$, cf. [1, pp. 127–128]. They are the statements saying that the first, the second separation and the reduction principles are true for a class Q, respectively.

9) Cf. [1, p. 133].

10) Cf. [7] and also [10].

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