17. *Note on Finite Semigroups which Satisfy Certain Group-like Condition*

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§1. **Introduction.** In this note we shall report promptly some results about $\mathfrak{S}$-semigroups and $\mathfrak{H}$-semigroups without proof. The propositions will be precisely discussed in another papers [3, 4].

A finite semigroup S is said to have $\mathfrak{S}$-property if S of order n contains no proper subsemigroup of order greater than $n/2$. We mean by a decomposition of S a classification of the elements into some classes due to a congruence relation. A decomposition is called homogeneous if each class is composed of equal number of elements. If every decomposition of a finite semigroup S is homogeneous, we say S has $\mathfrak{H}$-property, or S is called a $\mathfrak{H}$-semigroup.

According to Rees [1], if a finite semigroup S is simple, it is represented as a regular matrix semigroup with a ground group G and with a defining matrix $P=(p_{ji})$ of type (l, m), namely

either $\quad S=\{(x;\ i\ j)\mid x\in G,\ i=1,\cdots,m;\ j=1,\cdots,l\}$

or $\quad S=\{(x;\ i\ j)\mid x\in G,\ i=1,\cdots,m;\ j=1,\cdots,l\}\smile\{\mathbf{0}\}$

in which $\mathbf{0}$ is the two-sided zero of S. The multiplication is defined as

$$(x;\ i\ j)(y;\ s\ t)=\begin{cases}(xp_{js}y;\ i\ t) & \text{if } p_{js}\neq 0\\ \mathbf{0} & \text{if } p_{js}=0 \text{ and hence } S \text{ has } \mathbf{0}.\end{cases}$$

Let $M=\{1,\cdots,m\}$, $L=\{1,\cdots,l\}$. M and L are regarded as a right-singular semigroup and a left-singular semigroup respectively. For the sake of convenience, the notations

$$\text{Simp.}\,(G;P) \quad \text{and} \quad \text{Simp.}\,(G,0;P)$$

denote simple semigroups S with a ground group G and with a defining matrix P. The former is one without zero, whence $p_{ji}\neq 0$ for all i, j, but the latter denotes one with zero $\mathbf{0}$, so that if $p_{ji}\neq 0$ for all i and j, S contains no zero-divisor.

§2. **$\mathfrak{S}$-semigroups.** The following $\mathfrak{S}_1$-property is stronger than $\mathfrak{S}$-property, i.e. $\mathfrak{S}_1$-property implies $\mathfrak{S}$-property.

A finite semigroup S is said to have $\mathfrak{S}_1$-property if the order of any subsemigroup is a divisor of the order of S.

Let e be a unit of a finite group G.

Lemma 2.1. $\text{Simp.}\left(G;\begin{pmatrix}e\\e\end{pmatrix}\right)$ *is an $\mathfrak{S}_1$-semigroup.*

Lemma 2.1′. $\text{Simp.}\,(G;(e\ e))$ *is an $\mathfrak{S}_1$-semigroup.*

Lemma 2.2. *Let* $0 \neq a \in G$. Simp. $\left(G; \begin{pmatrix} e & e \\ e & a \end{pmatrix}\right)$ *is an* $\mathfrak{S}_1$*-semigroup.*

Lemma 2.3. *Let* $S=\text{Simp.}(G, 0; (p_{ji})\ i=1,\cdots,m;\ j=1,\cdots,l)$ *of order* >2. *S has no* $\mathfrak{S}$*-property.*

Lemma 2.4. *A finite non-simple semigroup has no* $\mathfrak{S}$*-property.*

Lemma 2.5. *Let* $S=\text{Simp.}(G; (p_{ji})\ i=1,\cdots,m;\ j=1,\cdots,l)$. *If S is an* $\mathfrak{S}$*-semigroup, then* $l \leqq 2$ *and* $m \leqq 2$.

Theorem 2.1. *A finite semigroup S is an* $\mathfrak{S}$*-semigroup of order* $\geqq 2$ *if and only if S is one of the following cases:*

(1) *a semilattice of order 2,*

(2) *a z-semigroup of order 2,*

(3) *a finite group of order* $\geqq 2$,

(4) Simp. $\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right)$,

(5) Simp. $(G; (e\ e))$,

(6) Simp. $\left(G; \begin{pmatrix} e & e \\ e & a \end{pmatrix}\right)$,

where e is a unit of G, $0 \neq a \in G$, *and the order of G is* $\geqq 1$.

Corollary 2.1. *A semigroup which contains no proper subsemigroup is either a semigroup of order at most 2 or a cyclic group of prime order* $\geqq 3$.

§3. **$\mathfrak{H}$-semigroups.** It goes without saying that any semigroup of order 2 and any indecomposable semigroup are $\mathfrak{H}$-semigroups.

Lemma 3.1. *A* $\mathfrak{H}$*-semigroup is simple and so completely simple.*

Lemma 3.2. *If a* $\mathfrak{H}$*-semigroup S has zero* **0**, *then S is an indecomposable semigroup* [2].

Corollary 3.1. *If a* $\mathfrak{H}$*-semigroup S has a non-trivial decomposition, then S is a simple semigroup without zero.*

Lemma 3.3. *Let* $S=\text{Simp.}(G; (p_{ji})\ i=1,\cdots,m;\ j=1,\cdots,l)$. *If S is a* $\mathfrak{H}$*-semigroup, then* $m \leqq 2$ *and* $l \leqq 2$. *Therefore S is a simple semigroup whose defining matrix is* $\begin{pmatrix} e \\ e \end{pmatrix}$ *or* $(e\ e)$ *or* $\begin{pmatrix} e & e \\ e & a \end{pmatrix}$ $a \neq 0$.

On the other hand we can prove

Lemma 3.4. Simp. $\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right)$, Simp. $(G; (e\ e))$ *and* Simp. $\left(G; \begin{pmatrix} e & e \\ e & a \end{pmatrix}\right)$ *are all* $\mathfrak{H}$*-semigroups.*

Thus we have

Theorem 3.1. *A finite semigroup of order* $\geqq 2$ *is a* $\mathfrak{H}$*-semigroup if and only if it is one of the following seven cases:*

(1) *a semilattice of order 2,*

(2) *a z-semigroup of order 2,*

(3) *an indecomposable finite semigroup of order* >1,

(4) *a finite group of order* $\geqq 2$,

(5) Simp. $\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right)$,

(6) Simp. $(G; (e\ e))$,

(7) Simp. $\left(G; \begin{pmatrix} e & e \\ e & a \end{pmatrix}\right)$,

where G *is a finite group of order* $\geqq 1$, e *is a unit, and* $a \neq 0$.

§4. **Remark.** As consequence of §2, we see that $\mathfrak{S}$-property implies $\mathfrak{S}_1$-property, that is, $\mathfrak{S}$-property and $\mathfrak{S}_1$-property are equivalent. Also, from the result of §3, it follows that $\mathfrak{S}$-property implies $\mathfrak{H}$-property; and moreover $\mathfrak{H}$-property implies $\mathfrak{S}$-property under the assumption that S is not an indecomposable semigroup.

References

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[3] ——: Finite semigroups in which Lagrange's theorem holds, Jour. Gakugei, Tokushima Univ., **10** (to be published).

[4] ——: Decompositions of a completely simple semigroup (to appear).