# 51. A Remark on a Paper of Greub and Rheinboldt

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(Comm. by K. KUNUGI, M.J.A., April 12, 1960)

1. In the first place, it will be shown by an elementary inspection the following

THEOREM 1. *For* $0<m<M$, *the following inequality holds true;*

(1) $$\int_m^M t\,d\mu(t)\cdot\int_m^M \frac{1}{t}\,d\mu(t) \leqq \frac{(M+m)^2}{4Mm},$$

*for any positive Stieltjes measure* $\mu$ *on* $[m, M]$ *with* $\|\mu\|=1$.

Consider a line-segment $C$ and a curve $D$ figured in $(t, s)$-plane by $(t, t)$ and $\left(t, \frac{1}{t}\right)$ respectively (for $m\leqq t\leqq M$). Putting

$$d=\int_m^M t\,d\mu(t), \qquad e=\int_m^M \frac{1}{t}\,d\mu(t),$$

$(d, d)$ is the centre of gravity of $C$ weighted by $\mu$, and $(d, e)$ is of $D$ weighted by the same $\mu$. Clearly, $(d, d)$ lies on $C$, and $(d, e)$ lies in the bow shaped territory bounded below by $D$ and above by its string connected $(m, 1/m)$ and $(M, 1/M)$ or the line figured by $(t, g(t))$ where

$$g(t)=\frac{(M+m)-t}{Mm}.$$

It is now obvious that the left hand side of (1), say $c$, is the product of the $s$-coordinates of two centres of gravity. Hence $(d, c)$ lies below a curve figured by $(t, h(t))$ with

$$h(t)=t\,g(t)=\frac{(M+m)t-t^2}{Mm}.$$

Therefore, $c$ amounts its maximum, if possible, when

$$Mm\,h'(t)=(M+m)-2t=0,$$

or $t=(M+m)/2$. Thus,

$$c\leqq h\left(\frac{M+m}{2}\right)=\frac{(M+m)^2}{4Mm},$$

which proves (1).

Incidentally, it is obvious that $c$ attains its maximum when

$$\mu(\{m\})=\mu(\{M\})=\frac{1}{2}.$$

THEOREM 2. *If* $f$ *is a continuous function defined on a compact set satisfying*

(2) $$0<m\leqq f(x)\leqq M,$$

*then*

$$\int_X f(x)d\mu \cdot \int_X \frac{1}{f(x)} d\mu \leqq \frac{(M+m)^2}{4Mm}, \tag{3}$$

*for any positive Borel measure $\mu$ with the total measure one.*

Since Theorem 2 is a verbal version of Theorem 1, the proof will be omitted here.

2. Recently W. Greub and W. Rheinboldt [2] proved, as a generalization of an inequality of Kantorovič, the following

THEOREM 3. *If $A$ is a self-adjoint operator defined on a Hilbert space satisfying*

$$0 < m \leqq A \leqq M, \tag{4}$$

*then for any vector $x$*

$$(Ax, x)(A^{-1}x, x) \leqq \frac{(M+m)^2}{4Mm}(x, x)^2. \tag{5}$$

It is easy to see by the Gelfand representation of the $C^*$-algebra generated by $A$ and the identity that Theorem 3 is implied by Theorem 1 or Theorem 2 (for the representations of operator algebras, cf. J. Dixmier [1]), since (4) implies (2) when $f$ corresponds to $A$ by the representation or since $A$ corresponds to $t$ on $[m, M]$ by the representation, and since $(Ax, x)$ defines a normalized measure on the spectrum for a normalized vector $x$. Also, conversely, it is not hard to see by the operator representation cannonically induced by a normalized measure $\mu$, that Theorem 1 is a consequence of Theorem 3, since $\mu$ is representable by $(Ax, x)$ for some $x$ with $\|x\|=1$. Hence, Theorems 1, 2 and 3 are mutally equivalent.

## References

[1] J. Dixmier: Les Algèbres d'Opérateurs dans l'Espace Hilbertien, Paris (1957).

[2] W. Greub and W. Rheinboldt: On a generalization of an inequality of L. V. Kantorovich, Proc. Amer. Math. Soc., **10**, 407–415 (1959).