97. *Finite-to-one Closed Mappings and Dimension. III*

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The method of proof employed in the previous note [5, Theorem 4] can be applied to new characterizations[1] of dimension of metric spaces by means of a sequence of coverings, which generalize the results due to J. Nagata [7, Theorem 3] and C. H. Dowker and W. Hurewicz [2], as follows.

Theorem 1. *In order that a topological space R be a metrizable space with* $\dim R$[2] $\leq n$ *it is necessary and sufficient that there exists a sequence of locally finite coverings* $\mathfrak{H}_i=\{H_\alpha;\alpha\in A_i\}$, $i=1,2,\cdots$, *of R which satisfies the following conditions.*

(1) $\overline{\mathfrak{H}}_{i+1}=\{\overline{H}_\alpha;\alpha\in A_{i+1}\}$ *refines* $\mathfrak{H}_i$ *for every i.*

(2) $\liminf_i \operatorname{order}(x,\mathfrak{H}_i)$[3] $\leq n+1$ *for every* $x\in R$.

(3) *For any point $x\in R$ and any neighborhood U of x there exists i with* $\operatorname{Star}(x,\mathfrak{H}_i)$[4] $\subset U$.

C. H. Dowker and W. Hurewicz's characterization [2] is a direct consequence of this theorem. As a corollary of this theorem we get the following.

Theorem 2. *In order that a topological space R be a metrizable space with $\dim R\leq n$ it is necessary and sufficient that there exists a sequence of open coverings $\mathfrak{U}_i$, $i=1,2,\cdots$, of R which satisfies the following conditions.*

(1) $\mathfrak{U}_{i+1}^*$[5] *refines* $\mathfrak{U}_i$ *for every i.*

(2) $\liminf_i \operatorname{order}(x,\mathfrak{U}_i)\leq n+1$ *for every* $x\in R$.

(3) *For any point $x\in R$ and any neighborhood U of x there exists i with* $\operatorname{Star}(x,\mathfrak{U}_i)\subset U$.

J. Nagata's characterization [7, Theorem 3] is a direct consequence of this theorem.

We call a covering U of a space a multiplicative[6] one if for every non-empty intersection $\bigcap_{i=1}^k U_i$ of elements U_i, $i=1,\cdots,k$, of $\mathfrak{U}$ is also an element of $\mathfrak{U}$. The maximal number n such that there

1) The detail of the content of the present note will be published in another place.

2) $\dim R$ denotes the covering dimension of R.

3) $\operatorname{order}(x,\mathfrak{H}_i)$ denotes the number of elements of $\mathfrak{H}_i$ which contain x.

4) $\operatorname{Star}(x,\mathfrak{H}_i)=\bigcup\{H_\alpha;x\in H_\alpha\in\mathfrak{H}_i\}$.

5) $\mathfrak{U}_{i+1}^*=\{\operatorname{Star}(U,\mathfrak{U}_{i+1});U\in\mathfrak{U}_{i+1}\}$, where $\operatorname{Star}(U,\mathfrak{U}_{i+1})=\bigcup\{V;U\cap V\neq\phi$ (=the empty set), $V\in\mathfrak{U}_{i+1}\}$.

6) Cf. [1] or [7].

exists a sequence $U_1 \supsetneqq U_2 \supsetneqq \cdots \supsetneqq U_n \neq \phi$ of elements of a multiplicative covering $\mathfrak{U}$ is called the length of $\mathfrak{U}$.

Theorem 3. *In order that a topological space R be a metrizable space with* $\dim R \leq n$ *it is necessary and sufficient that there exists a sequence of locally finite multiplicative coverings $\mathfrak{H}_i$, $i=1,2,\cdots$, of R which satisfies the following conditions.*

(1) $\overline{\mathfrak{H}}_{i+1}$ *refines* $\mathfrak{H}_i$ *for every* i.

(2) *The length of* $\mathfrak{H}_i \leq n+1$ *for every* i.

(3) *For any point* $x \in R$ *and any neighborhood* U *of* x *there exists* i *with* $\mathrm{Star}(x, \mathfrak{H}_i) \subset U$.

This theorem is to be compared with [7, Theorem 4].

We execute to construct a dimension-preserving completion of a metric space with the aid of our characterization theorems[7] as follows.

Theorem 4. *Let R be a metrizable space with* $\dim R \leq n$. *Let* $\mathfrak{U}_i$, $i=1,2,\cdots$, *be a sequence of open coverings of R which satisfies the following conditions.*[8]

(1) $\mathfrak{U}^*_{i+1}$ *refines* $\mathfrak{U}_i$ *for every* i.

(2) *Each element of* $\mathfrak{U}_{i+1}$ *intersects at most* $n+1$ *elements of* $\mathfrak{U}_i$ *for every* i.

(3) *For any point* $x \in R$ *and any neighborhood* U *of* x *there exists* i *with* $\mathrm{Star}(x, \mathfrak{U}_i) \subset U$.

Then a completion S of R with respect to $\mathfrak{U}_i$, $i=1,2,\cdots$, *has dimension* $\leq n$.

As an application of this theorem we get the following theorem, where J. Nagata's metric is a metric ρ with the following property:[9] When R is a metrizable space with $\dim R \leq n$, one can define a metric $\rho(x, y)$ agreeing with the topology of R such that for every $\varepsilon > 0$ and for any point $x \in R$, $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i=1,\cdots,n+2)$ imply $\rho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$, where $S_{\varepsilon/2}(x) = \{y; \rho(x, y) < \varepsilon/2\}$.

Theorem 5.[10] *Let R be a metrizable space with* $\dim R \leq n$. *Then a completion S of R with respect to J. Nagata's metric on R has dimension* $\leq n$.

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7) The existence of a dimension-preserving completion has already been proved by M. Katětov [3, Theorem 3.10] and by K. Morita [4, Theorem 5.6] independently.

8) The existence of a sequence with these conditions is guaranteed by [7, Theorem 2].

9) Cf. J. Nagata [7, Theorem 5].

10) This theorem can be proved with no use of Theorem 4 as follows: Let ρ be Nagata's metric on R and $\bar{\rho}$ a metric on S generated by ρ. Then it can be verified that $\bar{\rho}$ is also Nagata's metric on S. Hence get $\dim S \leq n$. This remark is given by Prof. Morita.

References

[1] P. Alexandroff und A. Kolmogoroff: Endliche Überdeckungen topologischer Räume, Fund. Math., **26**, 267–271 (1936).

[2] C. H. Dowker and W. Hurewicz: Dimension of metric spaces, Fund. Math., **43**, 83–88 (1956).

[3] M. Katětov: On the dimension of non-separable spaces I, Čehoslovack Mat. Ž., **2** (77), 333–368 (1952).

[4] K. Morita: Normal families and dimension theory for metric spaces, Math. Ann., **128**, 350–362 (1954).

[5] K. Nagami: Finite-to-one closed mappings and dimension. I, Proc. Japan Acad., **34**, 503–506 (1958).

[6] K. Nagami: Finite-to-one closed mappings and dimension. II, Proc. Japan Acad., **35**, 437–439 (1959).[11]

[7] J. Nagata: Note on dimension theory for metric spaces, Fund. Math., **45**, 143–181 (1958).

11) Let me take this opportunity of correcting my previous paper [6]: For '$\sup\dim R_\alpha$' in [6, Theorem 3] read '$\liminf_\alpha \dim R_\alpha$'.