# 96. A Necessary and Sufficient Condition under which $\dim(X\times Y)=\dim X+\dim Y$

By Yukihiro KODAMA

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§ 1. **Introduction.** Let $X$ and $Y$ be locally compact fully normal spaces. It is well known that the relation $\dim(X\times Y)\leqq\dim X+\dim Y$ holds, where dim means the covering dimension (cf. [12]). But, the following stronger relation (∗) does not hold in general:

(∗) $$\dim(X\times Y)=\dim X+\dim Y.$$

Some necessary conditions in order that the relation (∗) hold have been obtained by E. Dyer[1] and the author.[2] However, these conditions are not a sufficient condition.[3] The object of this paper is to obtain a necessary and sufficient condition under which the relation (∗) is true.

Let $G$ be an abelian group. The *homological dimension of* $X$ *with respect to* $G$ (notation: $D_*(X:G)$) is the largest integer $n$ such that there exists a pair $(A, B)$ of closed subsets of $X$ whose $n$-dimensional (unrestricted) Čech homology group $H_n(A, B:G)$[4] with coefficients in $G$ is not zero. A space $X$ is called *full-dimensional with respect to* $G$ if $D_*(X:G)=\dim X$. Let us use the following notations: $R$=the additive group of all rationals, $Z$=the additive group of all integers, $R_1$=the factor group $R/Z$, $Q_p$=the $p$-primary component of $R_1$ for a prime $p$, $Z_q$=the cyclic group with order $q(=Z/qZ)$, $Z(\mathfrak{a}_p)$=the limit group of the inverse system $\{Z_{p^i}:h_i^{i+1};\ i=1, 2,\cdots\}$, where $h_i^{i+1}$ is a natural homomorphism from $Z_{p^{i+1}}$ onto $Z_{p^i}$. We shall prove the following theorem.

**Theorem.** *Let $X$ and $Y$ be locally compact fully normal spaces. In order that the relation* $\dim(X\times Y)=\dim X+\dim Y$ *hold it is necessary and sufficient that at least one of the following four conditions be satisfied:*

(1) *$X$ and $Y$ are full-dimensional with respect to $R$.*
(2) *$X$ and $Y$ are full-dimensional with respect to $Z_p$ for a prime $p$.*
(3) *$X$ and $Y$ are full-dimensional with respect to $Z(\mathfrak{a}_p)$ and $Q_p$ for a prime $p$ respectively.*
(4) *$X$ and $Y$ are full-dimensional with respect to $Q_p$ and $Z(\mathfrak{a}_p)$ for a prime $p$ respectively.*

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1) Cf. [5, Theorem 4.1].
2) Cf. [10, Theorem 5].
3) Cf. [5, p. 141].
4) Cf. [4] and [9, p. 96].
5) Cf. [8, p. 385].

As Prof. K. Morita pointed out, there exists an intimate relation between the groups $Q_p$ and $R_p$,[6] where $R_p$ is the additive group of all rationals whose denominators are coprime with $p$. Therefore, in case $X$ and $Y$ are compact spaces, our theorem is a consequence of Bockstein's and Dyer's theorems (cf. [2], [3] and [5]). We shall give a simple and direct proof of the above theorem without making use of any duality and Bockstein's theorem.

## § 2. Lemmas

Let $X$ be a fully normal space (cf. [13]). By a covering we mean a locally finite open covering. Let $G$ and $H$ be abelian groups and $\rho$ a homomorphisms of $G$ into $H$. Let us define an auxiliary dimension function $d_*(X:\rho)$. Suppose that there exist a pair $(A, B)$ of closed subsets and a covering $\mathfrak{U}_0$ of $X$ having the following properties: whenever $\mathfrak{V}$ is any refinement of $\mathfrak{U}_0$, we have $0=\rho_*\Pi_*H_n(M, N:G)\subset H_n(K, L:H)$, where $(K, L)$ and $(M, N)$ are the pairs of the nerves of $\mathfrak{U}_0$ and $\mathfrak{V}$ corresponding to $(A, B)$, $\Pi_*$ and $\rho_*$ are natural homomorphisms induced by a projection $\Pi:(M, N)\to(K, L)$ and the homomorphism $\rho: G\to H$. Then we shall write $d_*(X:\rho)\geqq n$. If we have $d_*(X:\rho)\geqq n$, but not $d_*(X:\rho)\ngeqq i$ for any $i>n$, then we write $d_*(X:\rho)=n$. If $G=H$ and $\rho$ is the identity homomorphism, we shall write $d_*(X:G)$ instead of $d_*(X:\rho)$. Let us denote by $\rho_k[p]$ and $\rho_k^j[p]$ natural homomorphisms from $Z$ and $Z_{p^j}$ onto $Z_{p^k}$ respectively, where $k$ and $j$ are positive integers such that $j>k$.

**Lemma 1.** *An $n$-dimensional compact space $X$ is full-dimensional with respect to $R$ if and only if* $d_*(X:Z)=n$.

Proof. For any fully normal space $X$, the relations $d_*(X:R)=\dim X$ and $d_*(X:Z)=\dim X$ are equivalent. Since $R$ is a field, by [Lemma 5.8, Chap. VIII], the relation $D_*(X:R)=d_*(X:R)$ is true for any compact space $X$.

**Lemma 2.** *An $n$-dimensional compact space $X$ is full-dimensional with respect to $Z(\mathfrak{a}_p)$ if and only if $d_*(X:\rho_1[p])=n$ for the prime $p$.*

Proof. Let $U=\{\mathfrak{U}_\alpha \mid \alpha\in\Omega\}$ be a cofinal system of coverings of $X$ such that the order of each $\mathfrak{U}_\alpha$ is $n$. Suppose that $X$ is full-dimensional with respect to $Z(\mathfrak{a}_p)$. There exists a pair $(A, B)$ such that $0\neq H_n(A, B:Z(\mathfrak{a}_p))=\varprojlim\{H_n(K_\alpha, L_\alpha:Z(\mathfrak{a}_p)):\Pi^\beta_{\alpha*}\}$, where $(K_\alpha, L_\alpha)$ is the pair of the nerves of $\mathfrak{U}_\alpha$ and $\Pi^\beta_\alpha$ is a projection of $(K_\beta, L_\beta)$ into $(K_\alpha, L_\alpha)$ for $\alpha\leqq\beta\in\Omega$. Take a non-zero element $c=\{c_\alpha \mid \alpha\in\Omega\}$, where $c_\alpha$ is the $\alpha$-coordinate of the element $c$. Since $H_n(K_\alpha, L_\alpha:Z(\mathfrak{a}_p))\approx H_n(K_\alpha, L_\alpha:Z)\otimes Z(\mathfrak{a}_p)$, there exists an integral cycle $b_\alpha$ such that

6) Prof. K. Morita pointed out that the homological dimension with respect to $Q_p$ equals to the cohomological dimension with respect to $R_p$ for every compact space. See [10, footnote 1)].

$(\varphi_p)_* b_\alpha = c_\alpha$ for $\alpha \in \Omega$, where $\varphi_p$ is a natural homomorphism of $Z$ into $Z(\mathfrak{a}_p)$ defined by $\varphi_p(r) = \{\rho_k[p](r);\ k=1, 2, \cdots\}$ for $r \in Z$. Let $c_{\alpha_0} \neq 0$. Let $m$ be the positive integer such that the integral cycle $b_\beta$ is divisible by $p^{m-1}$ but not by $p^m$ for each $\beta \geqq \alpha_0$. Put $d_\beta = (1/p^{m-1}) \cdot b_\beta$, $\beta \geqq \alpha_0$. There exists an element $\beta_0$ of $\Omega$ such that $\beta_0 \geqq \alpha_0$ and $d_{\beta_0}$ is not divisible by $p$. Since $(\rho_1[p])_* (\Pi^\beta_{\beta_0})_* d_\beta = (\rho_1[p])_* d_{\beta_0} \neq 0$ for each $\beta \geqq \beta_0$, this shows that $\mathrm{d}_*(X : \rho_1[p]) = n$. Conversely, suppose that $\mathrm{d}_*(X : \rho_1[p]) = n$. There exist a pair $(A, B)$ of closed subsets and a covering $\mathfrak{U}_{\alpha_0}$ of $U$ such that $0 \neq (\rho_1[p])_* (\Pi^\beta_{\alpha_0})_* H_n(K_\beta, L_\beta : Z) \subset H_n(K_{\alpha_0}, L_{\alpha_0} : Z_p)$ for each $\beta \geqq \alpha_0$. The group $H_n(A, B : Z(\mathfrak{a}_p))$ is the limit group of the inverse system $\{H_n(K_\beta, L_\beta : Z_{p^k}) \mid (\rho^j_k[p])_* (\Pi^\beta_\alpha)_* : \alpha < \beta$ and $k < j\}$.[7] Since $(\rho^j_k[p])_* (\Pi^\beta_{\alpha_0})_* H_n(K_\beta, L_\beta : Z_{p^k})$ is a non-zero finite group for each $\beta \geqq \alpha_0$ and $k < j$, we can conclude that $H_n(A, B : Z(\mathfrak{a}_p)) \neq 0$. Thus, $X$ is full-dimensional with respect to $Z(\mathfrak{a}_p)$.

**Lemma 3.** *Let $X$ be an $n$-dimensional compact space. If $\mathrm{d}_*(X : Z) = n$, then $X$ is full-dimensional with respect to $Q_q$ for a prime $p$.*

**Lemma 4.** *If a compact space $X$ is full-dimensional with respect to $Z(\mathfrak{a}_p)$, then $X$ is full-dimensional with respect to $Q_p$.*

Since the relations $\mathrm{d}_*(X : Z(\mathfrak{a}_p)) = \dim X$ and $\mathrm{d}_*(X : Z) = \dim X$ are equivalent for any fully normal space $X$, it is sufficient to prove Lemma 3.[8] Let $\mathrm{d}_*(X : Z) = n$. Since each non-zero integral cycle is a non-zero cycle mod $p^k$ for some positive integer $k$, we have $\mathrm{d}_*(X : Q_p) = n$. Since $X$ is a compact space and $Q_p$ satisfies the descending chain condition, we can conclude that $X$ is full-dimensional with respect to $Q_p$.

**Lemma 5.** *An $n$-dimensional compact space $X$ is full-dimensional with respect to $Z_p$ if and only if $X$ is full-dimensional with respect to $Z_{p^k}$* for $k = 1, 2, \cdots$ .

The proof is obvious. The following lemma is well known [1].

**Lemma 6.** *Let $(K, L)$ be a pair of $n$-dimensional finite simplicial complexes. Then there exist integral cycles $Z_i$ and cycles $\Phi_j$* mod *powers of $p$, of $(K, L)$, such that the set $\{Z_i, \Phi_j\}$ forms a base of the group $H_n(K, L : Q_p)$; that is, for each cycle $c$ of $H_n(K, L : Q_p)$ there exists a linear combination $\sum t_i Z_i + \sum s_j \Phi_j$ which is congruent with $c$* mod 1, *where $t_i$ and $s_j$ are elements of $Q_p$.*

§ 3. **Proof of Theorem.** (I) The sufficiency. By [10, Theorem 6], there exists a compact subset $A$ of $X$ such that $\mathrm{D}_*(A : G) = \dim X$ in case $X$ is full-dimensional with respect to $G$, where $G$ is one of the groups $R$, $Z_p$, $Z(\mathfrak{a}_p)$ and $Q_p$. Thus, the proof is given by the same way as in the proof of [9, Theorem 1].

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7) Cf. [6, Theorem 5.1] or [8, Lemma 9] and [7].

8) By Lemmas 1-4 and Theorem, we can prove Theorem 5 of [10] directly without making use of other theorems of [10].

(II) The necessity. We shall show that, if none of the conditions (1)-(4) hold, we have $\dim(X\times Y)<\dim X+\dim Y$. By Hopf's extension theorem [11, Theorem 1], it is sufficient to prove that $H_{m+n}((A, B)\times(C, D): R_1)=0$ for each pair $(A, B)$ and $(C, D)$ of compact subsets of $X$ and $Y$. Let $U=\{\mathfrak{U}_\alpha|\alpha\in\Omega\}$ and $V=\{\mathfrak{V}_\gamma|\gamma\in\Gamma\}$ be cofinal systems of coverings of $X$ and $Y$ such that the orders of $\mathfrak{U}_\alpha$ and $\mathfrak{V}_\gamma$, $\alpha\in\Omega$ and $\gamma\in\Gamma$, are $m$ and $n$ respectively, where $m=\dim X$ and $n=\dim Y$. Let $(A, B)$ and $(C, D)$ be any pairs of compact subsets of $X$ and $Y$, $(K_\alpha, L_\alpha)$ and $(M_\gamma, N_\gamma)$ the pairs of the nerves of $\mathfrak{U}_\alpha$ and $\mathfrak{V}_\gamma$ corresponding to them, $\Pi^\beta_\alpha$: $(K_\beta, L_\beta)\to(K_\alpha, L_\alpha)$ and $\psi^\delta_\gamma$: $(M_\delta, N_\delta)\to(M_\gamma, N_\gamma)$ projections for $\alpha\leqq\beta\in\Omega$ and $\gamma\leqq\delta\in\Gamma$ respectively. Let $p$ be a prime. Fix any elements $\alpha_0$ and $\gamma_0$ of $\Omega$ and $\Gamma$. Let us denote by $G_{\alpha\gamma}$ the group $H_{m+n}((K_\alpha, L_\alpha)\times(M_\gamma, N_\gamma): Q_p)$ and by $\chi^{\beta\delta}_{\alpha\gamma}$ the homomorphism $(\Pi^\beta_\alpha\times\psi^\delta_\gamma)_*$ of $G_{\beta\delta}$ into $G_{\alpha\gamma}$ for $\alpha_0\leqq\alpha\leqq\beta$ and $\gamma_0\leqq\gamma\leqq\delta$. Each cycle of $G_{\alpha\gamma}$ is written in the form $\sum q^{\alpha\gamma}_{ij}(Z^\alpha_i\times W^\gamma_j)+\sum r^{\alpha\gamma}_{ik}(Z^\alpha_i\times\Theta^\gamma_k)+\sum s^{\alpha\gamma}_{hj}(\Phi^\alpha_h\times W^\gamma_j)+\sum t^{\alpha\gamma}_{hk}(\Phi^\alpha_h\times\Theta^\gamma_k)$, where $Z^\alpha_i$ and $W^\gamma_j$ are integral cycles, $\Phi^\alpha_h$ and $\Theta^\gamma_k$ are cycles mod powers of the prime $p$, $\{Z^\alpha_i, \Phi^\alpha_h\}$ and $\{W^\gamma_j, \Theta^\gamma_k\}$ are basis of $H_m(K_\alpha, L_\alpha: Q_p)$ and $H_n(M_\gamma, N_\gamma: Q_p)$ respectively (cf. Lemma 6), $q^{\alpha\gamma}_{ij}$, $r^{\alpha\gamma}_{ik}$, $s^{\alpha\gamma}_{hj}$ and $t^{\alpha\gamma}_{hk}$ are elements of the group $Q_p$. Since the condition (1) is not satisfied, by Lemma 1, there exists either (a) an element $\alpha_1$ of $\Omega$ such that $(\Pi^\alpha_{\alpha_0})_*H_m(K_\alpha, L_\alpha: Z)=0$ for any $\alpha\geqq\alpha_1$ or (b) an element $\gamma_1$ of $\Gamma$ such that $(\psi^\gamma_{\gamma_0})_*H_n(M_\gamma, N_\gamma: Z)=0$ for any $\gamma\geqq\gamma_1$. Let us assume that the case (a) holds. There is no loss of generality. Then, each element of the group $\chi^{\alpha_1\gamma}_{\alpha_0\gamma_0}G_{\alpha_1\gamma}$ for any $\gamma\geqq\gamma_0$ is written in the form $\chi^{\alpha_1\gamma}_{\alpha_0\gamma_0}(\sum s^{\alpha_1\gamma}_{hj}(\Phi^{\alpha_1}_h\times W^\gamma_j)+\sum t^{\alpha_1\gamma}_{hk}(\Phi^{\alpha_1}_h\times\Theta^\gamma_k))$. If $\Phi^{\alpha_1}_h$ is a cycle mod $p^a$, we have $p^a\cdot s^{\alpha_1\gamma}_{hj}\equiv 0$ mod 1 for each $\gamma$ and $j$. Let $b$ be the smallest positive integer such that $p^b.s^{\alpha_1\gamma}_{hj}\equiv 0$ mod 1 for each $h$, $j$ and $\gamma$. Since the condition (4) is not satisfied, $Y$ is not full-dimensional with respect to $Z(\mathfrak{a}_p)$ by Lemma 4. Therefore, by Lemma 2, there exists an element $\gamma_1$ of $\Gamma$ such that each element of the group $(\psi^{\gamma_1}_{\gamma_0})_*H_n(M_{\gamma_1}, N_{\gamma_1}: Z)$ is divisible by $p^b$. Thus, each element of the group $\chi^{\alpha\gamma}_{\alpha_1\gamma_1}G_{\alpha\gamma}$ is written in the form $\chi^{\alpha_1\gamma}_{\alpha_0\gamma_0}(\sum t^{\alpha_1\gamma}_{hk}(\Phi^{\alpha_1}_h\times\Theta^\gamma_k))$ for each $\alpha\geqq\alpha_1$ and $\gamma\geqq\gamma_1$. Since the condition (2) is not satisfied, at least one of the relations $\mathrm{D}_*(X: Z_p)<m$ and $\mathrm{D}_*(Y: Z_p)<n$ is true. Suppose that $\mathrm{D}_*(X: Z_p)<m$. Let $p^c$ be the order of the cycle $\Phi^{\alpha_1}_h$. Since the chain $t^{\alpha_1\gamma}_{hk}(\Phi^{\alpha_1}_h\times\Theta^\gamma_k)$ is a cycle mod 1, we have $p^c\cdot t^{\alpha_1\gamma}_{hk}\equiv 0$ mod 1 for each $k$ and $\gamma\geqq\gamma_1$. Since there exists only a finite number of $\Phi^{\alpha_1}_h$, we can find a positive integer $d$ such that $p^d\cdot t^{\alpha_1\gamma}_{hk}\equiv 0$ for each $h$, $k$ and $\gamma\geqq\gamma_1$. By Lemma 5, there exists an element $\alpha_2$ of $\Omega$ such that $0=(\Pi^{\alpha_2}_{\alpha_1})_*H_m(K_{\alpha_2}, L_{\alpha_2}: Z_{pd})\subset H_m(K_{\alpha_1}, L_{\alpha_1}: Z_{pd})$. Then the group $\chi^{\alpha_2\gamma_1}_{\alpha_0\gamma_0}(G_{\alpha_2\gamma_1})$ is zero. In case the relation $\mathrm{D}_*(Y: Z_p)<n$ is true, we can prove similarly that there exists an element $\gamma_2$ of $\Gamma$ such that $\chi^{\alpha_1\gamma_2}_{\alpha_0\gamma_0}(G_{\alpha_1\gamma_2})=0$. Since $\alpha_0$ and $\gamma_0$ are any elements of $\Omega$ and $\Gamma$, this

shows that $H_{m+n}((A, B)\times(C, D):Q_p)=0$. Since $R_1\approx\sum_p Q_p$, we can conclude that $H_{m+n}((A, B)\times(C, D):R_1)=0$. This completes the proof.

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