117. On the Maximum Principles of Second Order Elliptic Differential Equations

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The aim of this note is to extend the well-known maximum principle of E. Hopf[1] concerning the general second order elliptic differential equation

(1) $$F(x, u, u_k, u_{ij})=0,^{2)}$$

where $u_i=\partial u/\partial x_i$, $u_{ij}=\partial^2 u/\partial x_i \partial x_j$.

In this note we shall derive two kinds of the maximum principles under the following

Assumptions. I. The function $F(x, u, p_k, r_{ij})$ is defined in the domain $\mathfrak{D}: x \in G, |u|, |p_k|, |r_{ij}|<\infty$, where G is any domain in the Euclidean n-space.

II. $F(x, u, p_k, r_{ij})$ is continuously differentiable with respect to the arguments r_{ij} provided that the other arguments x, u, p_k remain fixed. Moreover, for every compact subset $\mathfrak{A}$ of $\mathfrak{D}$ there exists a constant $A>0$ such that

$$A^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j \leqq A|\xi|^2$$

for any $(x, u, p_i, r_{ij}) \in \mathfrak{A}$, and for any n-tuple $\xi=(\xi_1, \cdots, \xi_n)$.

III. $F(x, u, p_k, r_{ij})$ satisfies the Lipschitz condition with respect to the arguments u, p_i, r_{ij} in every compact subset of the domain $\mathfrak{D}$.

THEOREM I. *Let $u^{(1)}(x)$ and $u^{(2)}(x)$ be two $C^2(G)$-functions which satisfy the differential inequalities*

(2) $$F(x, u^{(1)}, u_k^{(1)}, u_{ij}^{(1)}) \leqq 0$$

and

(3) $$F(x, u^{(2)}, u_k^{(2)}, u_{ij}^{(2)}) \geqq 0$$

in the domain G respectively. We assume further that $u^{(2)}(x) \leqq u^{(1)}(x)$ in the domain G. Then we have the following alternative:

Either $u^{(2)}(x) \equiv u^{(1)}(x)$ *in the domain G,*

or $u^{(2)}(x) < u^{(1)}(x)$ *throughout in G.*

Proof. The proof will be carried out by reducing the theorem to the less general lemma.

LEMMA. *If the function F is of the form*

1) E. Hopf: Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, Sitzungsberichte Preuss. Akad. Wiss., **19**, 147-152 (1927).

2) We denote by x the point $(x_1, \cdots, x_n)$ of the Euclidean n-space.

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{ij}+f(x, u, u_k), \tag{4}$$

then the theorem holds.

As for the proof of this lemma the reader may refer to Prop. 9 of the author's previous note.[3]

Now, let us prove the theorem. According to Assumption II we can write

$$\begin{aligned} &F(x, u^{(1)}, u_k^{(1)}, u_{ij}^{(1)})-F(x, u^{(1)}, u_k^{(1)}, u_{ij}^{(2)}) \\ &\quad =\sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}}(x, u^{(1)}, u_k^{(1)}, v_{ts})(u_{ij}^{(1)}-u_{ij}^{(2)}), \end{aligned} \tag{5}$$

where $v_{ts}(x)$ are n^2 suitable functions which are of the form

$$u_{ts}^{(1)}(x)+\theta(x)(u_{ts}^{(2)}(x)-u_{ts}^{(1)}(x)), \quad 0<\theta(x)<1. \tag{6}$$

Next, we consider the elliptic differential equation

$$G(x, u, u_k, u_{ij})\equiv \sum_{i,j=1}^{n} a_{ij}(x)u_{ij}+g(x, u, u_k)=0, \tag{7}$$

where

$$a_{ij}(x)\equiv\frac{\partial F}{\partial r_{ij}}(x, u^{(1)}(x), u_k^{(1)}(x), v_{ts}(x)), \tag{8}$$

$$\begin{aligned} g(x, u, p_k)\equiv &F(x, u^{(1)}(x), u_k^{(1)}(x), u_{ij}^{(2)}(x)) \\ &-F(x, u^{(1)}(x)+u, u_k^{(1)}(x)+p_k, u_{ij}^{(2)}(x)). \end{aligned} \tag{9}$$

Clearly the function $G(x, u, p_k, r_{ij})$ satisfies Assumptions I–III. Hence we get the theorem, since the function

$$u\equiv u^{(1)}-u^{(2)} \tag{10}$$

satisfies the differential inequality

$$G(x, u, u_k, u_{ij})\leqq 0 \tag{11}$$

and since the identically zero function satisfies the differential equation

$$G(x, 0, 0, 0)=0. \tag{12}$$

To derive a maximum principle of E. Hopf's type we shall further impose the following additional

Assumptions. IV. The function $F(x, u, p_k, r_{ij})$ is non-increasing with respect to the argument u; i.e.

$$F(x, u, p_k, r_{ij})\leqq F(x, u', p_k, r_{ij}) \quad \text{provided } u\leqq u'.$$

V. The underlying domain G is bounded so that its closure $\overline{G}$ is a compact subset of the n-space.

THEOREM II. *Let $u^{(1)}(x)$ and $u^{(2)}(x)$ be two $C^2(G)\frown C(\overline{G})$-functions which satisfy the differential inequalities* (2) *and* (3) *in the domain G respectively. If $u^{(2)}(x)\leqq u^{(1)}(x)+\alpha$ on the boundary of G with a non-negative constant α, then the following alternative holds:*

3) K. Akô: On a generalization of Perron's method for solving the Dirichlet problem of second order partial differential equations, J. Fac. Sci. Univ. Tokyo, sec. I, **8**, 263–288 (1960). In the present note the continuity of the matrix $||a_{ij}(x)||$ is not required. But the proof of Prop. 9 remains valid in spite of this alteration.

Either $u^{(2)}(x) \equiv u^{(1)}(x) + \alpha$ *in the closure* $\overline{G}$ *of* G,
or $u^{(2)}(x) < u^{(1)}(x) + \alpha$ *throughout in* G.

Proof. Let β be the greatest value of the function $u^{(1)}(x) - u^{(2)}(x)$ in the closure $\overline{G}$ of G. If β is less than α the proof is completed. So we shall assume that $\beta \geqq \alpha$. Since the function $u^{(1)}(x) + \beta$ satisfies the differential inequality of the type (2) in G, and since the function $u^{(2)}(x)$ does the differential inequality (3) we get
either $u^{(2)}(x) \equiv u^{(1)}(x) + \beta$
or $u^{(2)}(x) < u^{(1)}(x) + \beta$
everywhere in the domain G. Therefore, we see that $\beta \leqq \alpha$ and hence $\beta = \alpha$. Thus we have established the theorem.

REMARK. The assumptions of Theorem II can be slightly modified as follows:

1°. The functions $u^{(1)}(x)$ and $u^{(2)}(x)$ are in $C^2(G)$ instead of being in $C^2(G) \frown C(\overline{G})$, where G is any bounded or unbounded domain.

2°. For every boundary point x of G

$$\liminf_{y \to x, y \in G} (u^{(1)}(y) - u^{(2)}(y)) \geqq -\alpha \quad (\alpha \geqq 0)$$

instead of the validity of the condition

$$u^{(2)}(x) \leqq u^{(1)}(x) + \alpha$$

on the boundary of G. Here, if G is unbounded the infinity ∞ must be considered to be a boundary point of G.

REMARK. The assumptions for Theorem II cannot essentially be modified. See the following

Example. $F \equiv \sum_{i=1}^{n} r_{ii} - 2k(2k+1) \cdot \sqrt[2k+1]{u^{2k-1}} \quad (k = 1, 2, \cdots)$,
G: any domain containing the origin.

We have two solutions $u_1(x) = 0$ and $u_2(x) = -|x_1|^{2k+1}$ of $F = 0$ for which the maximum principle (Theorem II) does not hold.