17. On Tonelli's Theorem concerning Curve Length

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1. **Introduction.** Let us consider a plane parametric curve (not necessarily continuous) given by the equation $\varphi(t)=\langle x(t), y(t)\rangle$, where the variable t ranges over the real line $\boldsymbol{R}$. We assume that this curve is locally rectifiable, i.e. that its arc length $s(I)$ is finite for any closed interval I in $\boldsymbol{R}$. We are interested in the problem of expressing the length by means of the derivatives $x'(t)$ and $y'(t)$. Of course this is easily solved when, in particular, the curve is continuously differentiable, since we have then, for every I, the well-known formula

$$s(I)=\int_I \sqrt{x'(t)^2+y'(t)^2}\,dt. \tag{1}$$

In the general case, however, the same problem shows itself far more complicated and was not solved until Tonelli proved the following decisive result: *we have the relation $s'(t)^2=x'(t)^2+y'(t)^2$ for almost every point t of $\boldsymbol{R}$, and the integral on the right of* (1) *does not exceed $s(I)$ for any closed interval I, the equality* (1) *holding if and only if both the functions $x(t)$ and $y(t)$ are absolutely continuous on I* (see Saks [3], p. 123).

Now Tonelli's theorem, though without doubt fautless in its own way, cannot nevertheless be regarded, so far as it goes, as a complete and final solution of the problem under consideration, in the following one point: it gives us no insight, even when the curve is continuous, into the nature of the difference between the arc length $s(I)$ and the square-root integral. It is the main object of the present note to remedy this defect by obtaining, at least for continuous curves, a supplement to Tonelli's theorem which resembles in enunciation the decomposition formula of de la Vallée Poussin (*vide* Saks, p. 127).

2. **Heuristic considerations.** Retaining the notation of the introduction, let us write E_x for the Borel set of the points t for which $x'(t)=\pm\infty$, and let E_y be defined correspondingly. According to de la Vallée Poussin's theorem (*loc. cit.*) we have, for every bounded Borel set A at whose points t the curve $\varphi(t)$ is continuous,

$$x^*(A)=x^*(AE_x)+\int_A x'(t)dt$$

and a similar relation for y^* (the set E_x being replaced by E_y, needless to say), where x^* and y^* represent the outer measures of Carathéodory induced by $x(t)$ and $y(t)$ respectively. This at once suggests us the conjecture that, if we write $E=E_x\cup E_y$ for brevity, then

$$s^*(M)=s^*(ME)+\int_M \sqrt{x'(t)^2+y'(t)^2}\,dt$$

for any bounded Borel set M at whose points $\varphi(t)$ is continuous. This decomposition formula will constitute a supplement to Tonelli's theorem since we have $s^*(M)=s(M)$ when M is in particular a closed interval. We shall prove this conjecture in § 4 in a somewhat generalized form.

With the view of application to differential geometry of parametric curves we shall also consider the case in which the curve φ is only unilaterally continuous at the points of the set M appearing above. But then the notion of derivative will be subjected to a slight corresponding modification in order to keep up the validity of the decomposition formula (see § 5).

3. Relative derivation of additive interval-functions. We shall henceforth follow our recent paper [2] in terminology and notation. Thus μ will always represent a finite nonnegative set-function, defined and additive on the class of all bounded Borel sets in $\boldsymbol{R}$. We shall say that μ is *continuous at a point* t of $\boldsymbol{R}$, if $\mu(\{t\})=0$. When this is the case for every point t, μ will simply be termed *continuous*. Needless to say, there is at most a countable infinity of points of discontinuity for μ. Further let $F(I)$ denote hereafter a finite additive interval-function, defined in any manner for all closed intervals I in $\boldsymbol{R}$ and of bounded variation over each I.

In Saks [3] the derivation of additive interval-functions is reduced to that of additive set-functions by means of the relation

$$\underline{D}F^*(t)\leqq\underline{F}(t)\leqq\overline{F}(t)\leqq\overline{D}F^*(t)$$

which holds for nonnegative F at every point of continuity of F. However, a similar reduction is no more available if we want to deal with relative derivation of additive interval-functions with respect to the function μ, since we do not assume the continuity of μ. We shall get over this situation by making the reduction on a different principle, basing our argument on the following lemma.

LEMMA. *At $\tilde{\mu}$-almost every point at which one at least of the functions μ and F is continuous, both the functions F and F^* are μ-derivable and their μ-derivatives coincide.*

REMARK. The proof below will be modelled after the proof of Lebesgue's theorem on p. 115 of Saks; but the details will be considerably different.

PROOF. Let X denote the set of the points of continuity of μ at which F is discontinuous. This set is then clearly countable and consequently we find at once that $\tilde{\mu}(X)=0$. Thus it is sufficient to restrict ourselves to points of continuity of F. Without loss of generality we may further assume F nonnegative. We shall write

Θ for F^* for simplicity.

Let A denote the Borel set of the points t which fulfil the inequality $(\mu)\overline{\Theta}(t) > (\mu)\underline{F}(t)$ and at which the function F is continuous. We want to show that $\tilde{\mu}(A)$ vanishes. For this purpose suppose, if possible, that $\tilde{\mu}(A) > 0$. We then find easily, as in Saks, that there exist a pair of positive numbers p, q and a bounded Borel set $B \subset A$ of positive measure (μ) in such a manner that, for all points t of B, we have $(\mu)\overline{\Theta}(t) > p > q > (\mu)\underline{F}(t)$.

Let ε be any positive number and G a bounded open set which contains B and satisfies both

(1) $$\mu(G-B) < q^{-1}\varepsilon \quad \text{and} \quad \Theta(G-B) < \varepsilon.$$

Consider the family of all closed intervals $I \subset G$ for which $F(I) \leqq q\mu(I)$. This family covers the set B in the Vitali sense, and therefore, in accordance with the covering theorem established in our paper [1], contains a disjoint (finite or infinite) sequence of intervals $I_1, I_2, \cdots$ whose join Q contains B almost entirely (μ). Now, if J° denotes the interior of an arbitrary closed interval J, then

$$\Theta(J) = \Theta(J^\circ) + \Theta(J - J^\circ) \leqq F(J) + \Theta(J - B),$$

since $\Theta(J^\circ) \leqq F(J)$ by a known theorem (Saks, p. 68, above) and since F, and hence Θ also, is continuous at every point of B. We thus find in view of (1) that

$$\Theta(Q) = \sum_n \Theta(I_n) \leqq q \sum_n \mu(I_n) + \sum_n \Theta(I_n - B) < q\mu(G) + \varepsilon < q\mu(B) + 2\varepsilon.$$

On the other hand, $(\mu)\overline{\Theta}(t) > p$ at each point t of B. Recalling the relation $\mu(B-Q)=0$ we therefore get, by the lemma of [2] § 4,

$$\Theta(Q) \geqq \Theta(BQ) \geqq p\mu(BQ) = p\mu(B).$$

This, in combination with what we have already proved, leads at once to $p\mu(B) < q\mu(B) + 2\varepsilon$, which is a contradiction since $p > q$, $\mu(B) > 0$, and ε is arbitrary. Consequently we must have $\tilde{\mu}(A) = 0$, or in other words, the inequality $(\mu)\overline{\Theta}(t) \leqq (\mu)\underline{F}(t)$ holds at $\tilde{\mu}$-almost every point t at which F is continuous.

Now, since F is nonnegative, $F(J) \leqq \Theta(J)$ for any closed interval J and so $(\mu)\overline{\Theta}(t) \geqq (\mu)\overline{F}(t) \geqq (\mu)\underline{F}(t)$ for every t. It follows that the μ-derivative $(\mu)F'(t)$ exists and coincides with $(\mu)\overline{\Theta}(t)$ at $\tilde{\mu}$-almost every point t of continuity of F. The function Θ being μ-derivable almost everywhere $(\tilde{\mu})$ as shown in [2] § 4, the assertion follows.

Theorem. *If F is nonnegative and X is a bounded Borel set at each of whose points one or both of the functions μ and F are continuous, then we have*

$$F^*(X) \geqq \int_X (\mu)F'(t)\,d\mu(t),$$

the sign of equality holding if, and only if, the outer measure F^* is absolutely continuous (μ) on X.

PROOF. In view of the above lemma this is an immediate consequence of Lebesgue's decomposition theorem of [2] § 7.

THEOREM. *Let us denote by W the absolute variation of the interval-function F and by C the set of the points of continuity of F. Then, for the set A of the points of C at which both the μ-derivatives of F and of F^* exist and coincide, we have*

$$\tilde{\mu}(C-A)=W^*(C-A)=0.$$

PROOF. Let Θ be short for F^* as above. Writing $\nu(X)=\mu(X)+W^*(X)$ for bounded Borel sets X, so that ν is a nonnegative additive set-function, we see easily that $\tilde{\nu}(Y)=\tilde{\mu}(Y)+W^*(Y)$ for any set Y (cf. the final remark of [2] § 2 which concerns the construction of $\tilde{\mu}$ from μ). Let P be the set of the points of C at which the three functions μ, F, and Θ are all ν-derivable and moreover the ν-derivatives of F and of Θ coincide. Then $\tilde{\nu}(C-P)$ vanishes on account of the above lemma and Lebesgue's theorem of [2] § 4. Further let Q be the set of the points t of P at each of which $(\nu)\mu'(t)$ and $(\nu)F'(t)$ do not both vanish. Since then

$$(\mu)F'(t)=(\nu)F'(t)/(\nu)\mu'(t)=(\nu)\Theta'(t)/(\nu)\mu'(t)=(\mu)\Theta'(t)$$

at all points t of Q, we have $Q\subset A$, and this conjointly with $\tilde{\nu}(C-P)=0$ obtained above implies that

$$\tilde{\nu}(C-A)\leqq\tilde{\nu}(C-Q)\leqq\tilde{\nu}(C-P)+\tilde{\nu}(P-Q)=\tilde{\nu}(P-Q).$$

Thus the assertion will be established if we show that $\tilde{\mu}(P-Q)=0$ and $W^*(P-Q)=0$.

Now both the ν-derivatives $(\nu)\mu'(t)$ and $(\nu)\Theta'(t)$ vanish everywhere in $P-Q$. Consequently it follows from Corollary 1° of [2] § 7 that $\mu(Z)=0=\Theta(Z)$, where and subsequently Z represents an arbitrary bounded Borel set contained in $P-Q$. From the former of these equalities we deduce at once that $\tilde{\mu}(P-Q)=0$. On the other hand, the set Z being arbitrary, the latter equality plainly implies that, if Γ denotes the absolute variation of the restriction of the set-function Θ to bounded Borel sets, then $\Gamma(Z)=0$ for every Z. But it is known that $\Gamma(X)=W^*(X)$ for bounded Borel sets $X\subset C$ (see Saks, p. 99). Since every Z lies in C, it follows that $W^*(P-Q)=0$, which together with $\tilde{\mu}(P-Q)=0$ completes the proof, as already observed.

4. Supplement to Tonelli's theorem. Let us state firstly a version of Tonelli's theorem which can be proved in almost the same way as in Saks, pp. 124–125. The letter μ will retain its meaning explained in the foregoing section.

TONELLI'S THEOREM. *Given a locally rectifiable curve $\varphi(t)=\langle x(t), y(t)\rangle$ let $s(I)$ denote its arc length over any closed interval I and let M be the set of the points at each of which one or both of the func-*

tions μ *and* $\varphi(t)$ *are continuous. Then*

(1) $$[(\mu)s'(t)]^2=[(\mu)x'(t)]^2+[(\mu)y'(t)]^2$$

for $\tilde{\mu}$*-almost every point* t *of the set* M *and*

(2) $$s^*(X)\geqq\int_X\sqrt{[(\mu)x'(t)]^2+[(\mu)y'(t)]^2}\,d\mu(t)$$

for each bounded Borel set $X\subset M$. *The equality sign holds in* (2) *if, and only if, both the outer measures* P^* *and* Q^* *are absolutely continuous* (μ) *on the set* X, *where* P *and* Q *stand for the absolute variations of* $x(t)$ *and* $y(t)$ *respectively.*

We shall now complete this theorem by establishing the following supplement to it, which constitutes the main result of this note:

SUPPLEMENT. *Let* C *be the set of the points at which the curve* $\varphi(t)$ *of the above theorem is continuous, and* E *the Borel set of the points* t *of* C *at each of which one or both of the* μ*-derivatives* $(\mu)x'(t)$ *and* $(\mu)y'(t)$ *exist and are infinite. Then, for every bounded Borel set* $X\subset C$,

(3) $$s^*(X)=s^*(XE)+\int_X\sqrt{[(\mu)x'(t)]^2+[(\mu)y'(t)]^2}\,d\mu(t).$$

PROOF. Let us denote by K the Borel set of the points t of C at which $(\mu)s'(t)$ is infinite, and by N the Borel set of the points of K at which both the μ-derivatives of x and of x^* exist and coincide. Then K clearly contains E, and the function P of the above theorem fulfils $P^*(K-N)=0$ on account of the last theorem of §3. Thus

(4) $$P^*(K-E)\leqq P^*(K-N)+P^*(N-E)=P^*(N-E).$$

Now, since C plainly coincides with the set of the points of continuity of $s(t)$, the lemma of §3 implies that $\tilde{\mu}(K)=0$, from which we derive $\tilde{\mu}(N-E)=0$. On the other hand, the μ-derivative of x^* must be finite at any point t of $N-E$ since it is equal to $(\mu)x'(t)$, which cannot be infinite since t does not belong to E. Consequently, appealing successively to the formula (2) of [2] §8 and the theorem on p. 99 of Saks, we find at once that $P^*(Y)=0$ for all bounded Borel sets $Y\subset N-E$ and that therefore $P^*(N-E)=0$. This combined with (4) yields $P^*(K-E)=0$. By symmetry $Q^*(K-E)$ must also vanish, and we get finally $s^*(K-E)=0$ since evidently

$$s^*(K-E)\leqq P^*(K-E)+Q^*(K-E).$$

This being so, consider the Borel set K_0 of the points t of C at which the μ-derivative of s^* becomes $+\infty$. Since $s(I)\leqq s^*(I)$ for every closed interval I, the set K_0 contains K; and it follows from the last theorem of the preceding section that $s^*(K_0-K)=0$. This, in combination with $s^*(K-E)=0$ obtained already, gives $s^*(K_0-E)=0$, where we observe that $E\subset K_0$.

With this in mind we deduce from the decomposition formula (1) of [2] §8 that, for every bounded Borel set $X\subset C$,

$$s^*(XE)=s^*(XK_0)=s^*(X)-\int_X \frac{ds^*}{d\mu}(t)d\mu(t).$$

But the μ-derivative of s^* is equal to that of s at $\tilde{\mu}$-almost every point of C, in conformity with the lemma of § 3. The formula (3) follows now at once on account of the relation (1).

5. **Relative interior-derivative of an additive interval-function.** Consider a function T which is defined at least for all closed intervals in $\boldsymbol{R}$ and assumes finite values for such intervals. By the *interior μ-derivative* of T at a point t we shall understand the limit, supposed existent, of the ratio $T(I)/\mu(I)$ as $|I|$ tends to zero, where I stands for any closed interval whose interior contains the point t. We shall write $(\mu)T^{\mathfrak{c}}(t)$ for this quantity. As is easily seen, the set of the points at which $(\mu)T^{\mathfrak{c}}(t)$ exists is a Borel set and $(\mu)T^{\mathfrak{c}}(t)$ is B-measurable on this set (cf. Saks, p. 113, above). Plainly $(\mu)T^{\mathfrak{c}}(t)$ exists and equals $(\mu)T'(t)$ wherever the latter exists, but the converse is of course false. The function T will be termed *interior-derivable* (μ) at a point t when $(\mu)T(t)$ exists and is finite.

Let F be as in § 3 an additive interval-function, of bounded variation over closed intervals. *Then F is interior-derivable (μ) almost everywhere ($\tilde{\mu}$).* Indeed we easily find, for each point t at which not both $F^*(\{t\})$ and $\mu(\{t\})$ vanish, the relation

$$(\mu)F^{\mathfrak{c}}(t)=F^*(\{t\})/\mu(\{t\}),$$

and the assertion is a direct consequence of this and the lemma of § 3.

Let us return to the consideration of the preceding section, retaining the notation used there. We can now state the following theorem, whose proof will be given in a separate paper.

THEOREM. *Let C_0 be the set of the points t at which the curve $\varphi(t)$ is unilaterally continuous, and E_0 the Borel set of the points t at each of which one at least of $(\mu)x^{\mathfrak{c}}(t)$ and $(\mu)y^{\mathfrak{c}}(t)$ exists and is infinite. Then, for any bounded Borel set $X\subset C_0$,*

$$s^*(X)=s^*(XE_0)+\int_X \sqrt{[(\mu)x^{\mathfrak{c}}(t)]^2+[(\mu)y^{\mathfrak{c}}(t)]^2}\, d\mu(t).$$

REMARK. Of course φ need not be continuous on the same side at all points of the set C_0.

References

[1] Ka. Iseki: On the covering theorem of Vitali, Proc. Japan Acad., **36**, 539–542 (1960).

[2] Ka. Iseki: On relative derivation of additive set-functions, Proc. Japan Acad., **36**, 630–635 (1960).

[3] S. Saks: Theory of the Integral, Warszawa-Lwów (1937).