35. A Certain Type of Vector Field. III

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The objective of the present paper is to prove the theorem announced in section V of the preceding paper [1].

To avoid the trivial repetition of the same technique of proving we shall verify only the fact that the existence of a vector field (12) of the above place is equivalent to that of such a conformal separability as this:

(1) $$ds^2 = \sinh^2(cx^n+d)\, ds_0^2 + (dx^n)^2$$

where c and d are constants, and ds_0^2 is an $(n-1)$-dimensional metric form independent of x^n.

First let us assume that the metric form is conformally separable in the way of (1). Set $\xi_i = \delta_i^n$, where δ_i^n is the so-called Kronecker's delta. Then we have

$$\xi_{i|j} = c \coth(cx^n+d)\, g_{ij} - c \coth(cx^n+d)\, \xi_i \xi_j.$$

Let $V_i = \tanh(cx^n+d)$ and we get

$$\begin{aligned} V_{i|j} &= \tanh(cx^n+d)\, \xi_{i|j} + c\, \mathrm{sech}^2(cx^n+d)\, \xi_i \delta_j^n \\ &= c\, g_{ij} - c\{1-\mathrm{sech}^2(cx^n+d)\}\xi_i\xi_j \\ &= c\, g_{ij} - c \tanh^2(cx^n+d)\xi_i\xi_j = c(g_{ij} - V_iV_j). \end{aligned}$$

The converse is as follows. Suppose that V satisfies (12) of [1]. Then we have

(2) $$\tfrac{1}{2}(\|V\|^2)_{|j} = c(1-\|V\|)^2 V_j.$$

Taking a canonical coordinate to V, we have

$$V^i = \|V\|^2 \delta_n^i \quad \text{and} \quad g_{nn} = \frac{1}{\|V\|^2}.$$

From (2) we get

$$\tfrac{1}{2}(\|V\|^2)_{|n} = c(1-\|V\|^2).$$

Consequently

$$\frac{\|V\|_{|n}}{1-\|V\|^2} = c\sqrt{g_{nn}}.$$

Hence we find

(3) $$\|V\| = \tanh(cs+d),$$

where s is the arc length of the tangent curve. It is easily seen that d is a constant. From (10) of [1] we have

$$\begin{aligned} H(x) &= \exp 2\int \frac{c}{\tanh(cs+d)} \sqrt{g_{nn}}\, dx^n \\ &= \exp 2c \int \frac{ds}{\tanh(cs+d)} = \sinh^2(cs+d). \end{aligned}$$

Consequently $ds^2 = sinh^2(cs+d)\, ds_0^2 + \{F(x^n)\, dx^n\}^2$. This completes the proof of Theorem B.

The following theorem is easily seen.

Theorem F. *The manifold is of constant curvature if and only if one of the types of vector field stated in Theorem B exists in every direction, namely* (12), (13), *or* (14) *satisfies the condition of integrability.*

Remark. In each case of Theorem B, $H(x)$ necessarily vanishes at a certain point as is shown from properties of the trigonometric functions, if the manifold is complete. Hence the manifold becomes Riemannian (see [2]). This means that the theorem having been just proved is of Riemann from the global point of view, though it holds in the Finsler manifold under the condition $V \in S_x$.

References

[1] T. Maebashi: A certain type of vector field. II, Proc. Japan Acad., **37**, 137–141 (1961).
[2] T. Maebashi: Vector fields and space forms, J. Fac. Sci., **15**, 62–92 (1960).