32. A Note on Hausdorff Spaces with the Star-finite Property. I

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K. Morita [4, Theorem 10.3] proved that (α) every metric space with the star-finite property can be embedded into the product of a 0-dimensional metric space and the Hilbert fundamental cube.[1] Yu. M. Smirnov [5] proved the following theorem which is an immediate corollary of Morita's theorem cited now and seems to be probably obtained independently of Morita's work: (β) For every metric space R with the star-finite property there exist a 0-dimensional metric space S and a continuous mapping f of R onto S such that $f^{-1}(x)$ is separable for every point x of S.

The purpose of this note is to give an analogous proposition to (β) as follows.

Theorem 1. *Let R be a non-empty Hausdorff space with the star-finite property.*[2] *Then there exist a paracompact Hausdorff space A with* $\dim A$[3]$=0$ *and a continuous mapping f of R onto A such that for every point y of A $f^{-1}(y)$ has the Lindelöf property.*[4]

In view of Morita's theorem [2] we may expect that the condition imposed on f in our Theorem 1 will be strengthened to be closed: But it is, in general, impossible as Yu. Smirnov's example [5] shows. It seems to be difficult to obtain a refinement of Theorem 1 in an analogous expression to the proposition (α), because of the difficulty to get the space for our case which plays the same rôle as the Hilbert fundamental cube does for the metric spaces with the star-finite property.

To prove Theorem 1 let us start with finding the universal 0-dimensional paracompact Hausdorff spaces.

Definition. Let Λ be a directed set and $\{A_\lambda, f_{\lambda\mu}; \mu<\lambda, \mu, \lambda\in\Lambda\}$ be an inverse limiting system consisting of discrete spaces A_λ, where

1) This theorem has been improved by himself as follows: Every metric space with an open basis which is the sum of a countable number of star-countable open coverings can be embedded into the product of a 0-dimensional metric space and the Hilbert fundamental cube. An open covering is called star-countable if every element of it intersects at most countable elements of it.

2) An open covering of a topological space is called star-finite if every element of it intersects at most finite elements of it. According to Morita [3] a topological space is called to have the star-finite property if every open covering of it can be refined by a star-finite open covering.

3) $\dim A$ denotes the covering dimension of A.

4) A topological space is called to have the Lindelöf property if every open covering of it has a countable subcovering. Morita [3] proved that every regular space with the Lindelöf property has the star-finite property.

$f_{\lambda\mu}$ is a mapping of A_λ into A_μ. Let π_λ be the projection of $\lim\{A_\lambda, f_{\lambda\mu}\}$ into A_λ. We call the system $\{A_\lambda; f_{\lambda\mu}\}$ *full* if every open covering of $\lim\{A_\lambda, f_{\lambda\mu}\}$ can be refined by $\{\pi_\lambda^{-1}(\alpha_\lambda); \alpha_\lambda \in A_\lambda\}$ for some $\lambda \in \Lambda$.

Theorem 2. *In order that a topological space R be a paracompact Hausdorff space with* $\dim R=0$ *it is necessary and sufficient that R is homeomorphic to the non-empty limit space obtained from an inverse limiting full system which consists of discrete spaces.*

Proof. It is evident that the condition is sufficient. To prove the necessity let R be a paracompact Hausdorff space with $\dim R=0$. Let $\{\mathfrak{U}_\lambda=\{U_\alpha; \alpha\in A_\lambda\}; \lambda\in\Lambda\}$ be a system of all open coverings of R whose orders[5] are one. Let us introduce into Λ the semi-order $\lambda<\mu$ if and only if $\mathfrak{U}_\mu$ refines $\mathfrak{U}_\lambda$. Then Λ becomes a directed set. Consider A_λ as topological spaces with the discrete topology. Define $f_{\mu\lambda}: A_\mu\to A_\lambda(\lambda<\mu)$ in such a way that $f_{\mu\lambda}(\alpha)=\beta$ if and only if $U_\alpha\subset U_\beta$. Then $f_{\mu\lambda}$ is a mapping of A_μ onto A_λ and $\{A_\lambda, f_{\lambda\mu}; \lambda\in\Lambda\}$ becomes an inverse limiting system of discrete spaces. Let x be an arbitrary point of R. For every $\lambda\in\Lambda$ we can choose an index $x_\lambda\in A_\lambda$ with $x\in U_{x_\lambda}$. Then $(x_\lambda; \lambda\in\Lambda)$ is a point of $A=\lim\{A_\lambda, f_{\lambda\mu}\}$. Hence A is non-empty. Since for every $x\in R$ $(x_\lambda; \lambda\in\Lambda)$ is unique, we can define a mapping $f: R\to A$ in such a way that $f(x)=(x_\lambda)\in A$. We can prove that f gives a homeomorphism of R onto A as follows.

To show that f is onto let a be an arbitrary point of A. Then $\mathfrak{U}=\{U_{\pi_\lambda(a)}; \lambda\in\Lambda\}$ forms a basis of a Cauchy filtre with respect to the uniformity $\{\mathfrak{U}_\lambda; \lambda\in\Lambda\}$.[6] Since it is almost evident that a paracompact Hausdorff space is complete with respect to its universal (i.e. finest) uniformity and $\{\mathfrak{U}_\lambda; \lambda\in\Lambda\}$ is actually universal, $\mathfrak{U}$ converges to a point x of R. Since for every $\lambda\in\Lambda$ $U_{\pi_\lambda(a)}$ is open and closed, x belongs to $U_{\pi_\lambda(a)}$ for every $\lambda\in\Lambda$. Hence $f(x)=a$ and we know that f is onto.

Since $\{\mathfrak{U}_\lambda; \lambda\in\Lambda\}$ agrees with the topology of R, f is one-to-one. Let x be an arbitrary point of R and $f(x)=a$. Then $\{U_{\pi_\lambda(a)}; \lambda\in\Lambda\}$ and $\{\pi_\lambda^{-1}(\pi_\lambda(a))\}$ form complete neighborhood systems of x and a respectively. Moreover it is evident that $f(U_{\pi_\lambda(a)})=\pi_\lambda^{-1}(\pi_\lambda(a))$. Hence we know that both f and f^{-1} are continuous. Therefore f is a homeomorphic mapping.

The rest of the proof is to show that the system $\{A_\lambda, f_{\lambda\mu}\}$ is full. Let $\mathfrak{B}$ be an arbitrary open covering of A. Then there exists $\lambda\in\Lambda$ such that $\mathfrak{U}_\lambda$ refines $\{f^{-1}(V); V\in\mathfrak{B}\}$. Since $f(\mathfrak{U}_\lambda)=\{f(U); U\in\mathfrak{U}_\lambda\}$ refines $\mathfrak{B}$ and $f(U_\alpha)=\pi_\lambda^{-1}(\alpha)$ for every $\alpha\in A_\lambda$, $\{\pi_\lambda^{-1}(\alpha); \alpha\in A_\lambda\}$ refines $\mathfrak{B}$. Thus the system $\{A_\lambda\}$ is full and the theorem is proved.

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5) The order of an open covering $\mathfrak{U}$ is the supremum of the number of elements of $\mathfrak{U}$ with the non-empty intersection.

6) A uniformity in this note means a basis of a uniformity in the sense of J. W. Tukey [6].

The following proposition, which is well known, is an immediate consequence of this theorem.

Corollary 1. *For a topological space R the following conditions are equivalent.*

(1) *R is a compact Hausdorff space with* $\dim R=0$.

(2) *R is homeomorphic to a non-empty closed subset of the product of discrete spaces consisting of finite points.*

(3) *R is homeomorphic to the non-empty limit space of an inverse limiting system of discrete spaces consisting of finite points.*

Corollary 2.[7] *For a topological space R the following conditions are equivalent.*

(1) $\operatorname{ind} R$[8] $=0$.

(2) *R is homeomorphic to a non-empty dense subset of the limit space of an inverse limiting system of discrete spaces.*

(3) *R is homeomorphic to a non-empty dense subset of a compact Hausdorff space S with* $\dim S=0$.

Proof. (3) $\rightarrow$ (2) is obvious from Corollary 1. (2) $\rightarrow$ (1) is clear. Let $\operatorname{ind} R=0$. Then the family $\{\mathfrak{U}_\lambda;\ \lambda\in\Lambda\}$ of all finite open coverings of R whose orders are one forms a uniformity of R which agrees with its topology. Then an analogous argument to the proof of Theorem 2 yields at once the implication (1)$\rightarrow$(3).

Proof of Theorem 1. Let $\{\mathfrak{U}_\lambda=\{U_\alpha;\ \alpha\in A_\lambda\};\ \lambda\in\Lambda\}$ be a system of all open coverings of R whose orders are one. Starting from $\{\mathfrak{U}_\lambda\}$, we can construct A and f by the quite analogous method in the proof of Theorem 2. Define the semi-order $\lambda<\mu$ if and only if $\mathfrak{U}_\mu$ refines $\mathfrak{U}_\lambda$. Then Λ becomes a directed set. Consider A_λ as topological spaces with the discrete topology. Define $f_{\mu\lambda}: A_\mu\rightarrow A_\lambda(\lambda<\mu)$ in such a way that $f_{\mu\lambda}(\alpha)=\beta$ if and only if $U_\alpha\subset U_\beta$. Then $f_{\mu\lambda}$ is onto and $\{A_\lambda, f_{\lambda\mu}\}$ becomes an inverse limiting system of discrete spaces. Let x be an arbitrary point of R. For every $\lambda\in\Lambda$ choose an index $x_\lambda\in A_\lambda$ with $x\in U_{x_\lambda}$. Then evidently $(x_\lambda;\ \lambda\in\Lambda)$ is a point of $A=\lim\{A_\lambda, f_{\lambda\mu}\}$. Hence A is non-empty. Since for every $x\in R$ $(x_\lambda;\ \lambda\in\Lambda)$ is unique, we can define a mapping $f: R\rightarrow A$ in such a way that $f(x)=(x_\lambda)\in A$.

To show the fullness of $\{A_\lambda, f_{\lambda\mu}\}$ let $\mathfrak{U}$ be an arbitrary open covering of A. Since $\operatorname{ind} A=0$, $\mathfrak{U}$ can be refined by a covering $\mathfrak{B}$ whose elements are open and closed. Then $f^{-1}(\mathfrak{B})=\{f^{-1}(V);\ V\in\mathfrak{B}\}=\{W_\xi;\ \xi\in X\}$ is a covering of R whose elements are open and closed. Let $\mathfrak{D}=\{D_\eta;$

7) It is provable that many mathematicians have become aware of the validity of this corollary. The equivalency of (1) and (3) was proved for the first time by N. Vedenisoff whose proof is not available in our country.

It is to be noted that a topological space R with $\operatorname{ind} R=0$ is completely regular. As for an example of a completely regular space R with $\operatorname{ind} R=0$ which fails to be normal, see [1, Appendix].

8) $\operatorname{ind} R$ denotes the small inductive dimension of R. $\operatorname{ind} R=0$ if and only if there exists a basis of R consisting of open and closed sets.

$\eta\in Y\}$ be a star-finite open covering of R which refines $f^{-1}(\mathfrak{V})$. Then Y can be decomposed into mutually disjoint subsets Y_γ, $\gamma\in\Gamma$, such that i) Y_γ consists of countable indices for every $\gamma\in\Gamma$ and ii) $D_{\eta_1}\frown D_{\eta_2}=\phi$ whenever $\eta_1\in Y_{\gamma_1}$, $\eta_2\in Y_{\gamma_2}$ and $\gamma_1\neq\gamma_2$. Define a mapping $\varphi: Y\to X$ in such a way that $\varphi(\eta)=\xi$ yields $D_\eta\subset W_\xi$. Let $Y_\gamma=\{1(\gamma), 2(\gamma), \cdots\}$ and $E_{i(\gamma)}=W_{\varphi(i(\gamma))}\frown(\smile\{D_\eta;\ \eta\in Y_\gamma\})$, $i=1,2,\cdots$. Then $E_{i(\gamma)}$ is open and closed for any i and $\gamma\in\Gamma$. Let $E_{0(\gamma)}=\phi$ for any $\gamma\in\Gamma$. Then it can easily be seen that $\mathfrak{E}=\{E_{i(\gamma)}-\smile_{j<i}E_{j(\gamma)};\ i=1,2,\cdots,\gamma\in\Gamma\}$ is an open covering of R whose order is one. Thus $\mathfrak{E}=\mathfrak{U}_\lambda$ for some $\lambda\in\Lambda$. Since $\mathfrak{E}$ refines $f^{-1}(\mathfrak{V})$ and $\mathfrak{U}_\lambda=\{f^{-1}(\pi_\lambda^{-1}(\alpha));\ \alpha\in A_\lambda\}$, we know that $\{\pi_\lambda^{-1}(\alpha);\ \alpha\in A_\lambda\}$ refines $\mathfrak{V}$. Thus the fullness of $\{A_\lambda, f_{\lambda\mu}\}$ has established. By Theorem 2 A is therefore a paracompact Hausdorff space with $\dim A=0$.

The above argument proves essentially the following fact: For any relatively open covering $\mathfrak{W}$ of $f(R)$ there exists an index $\lambda\in\Lambda$ such that $\{f(R)\frown\pi_\lambda^{-1}(\alpha);\ \alpha\in A_\lambda\}$ refines $\mathfrak{W}$. Thus we know that $f(R)$ is paracompact and $S_1=\{\{f(R)\frown\pi_\lambda^{-1}(\alpha);\ \alpha\in A_\lambda\};\ \lambda\in\Lambda\}$ forms the universal structure of $f(R)$. Since i) $\overline{f(R)}=A$, ii) $f(R)$ is complete with respect to its universal structure, iii) S_1 is the restriction of the universal structure S of A, it is easy to verify that $f(R)=A$.

Let y be an arbitrary point of A. To show that $f^{-1}(y)$ has the Lindelöf property, let $\mathfrak{G}$ be an arbitrary relatively open covering of $f^{-1}(y)$. Since $\mathfrak{G}_1=\{(R-f^{-1}(y))\smile G;\ G\in\mathfrak{G}\}$ is an open covering of R, there exists a star-finite open covering $\mathfrak{H}=\{H_\delta;\ \delta\in\Delta\}$ of R which refines $\mathfrak{G}_1$. Δ can be decomposed into mutually disjoint subsets Δ_θ, $\theta\in\Theta$, such that i) Δ_θ consists of countable indices for every $\theta\in\Theta$ and ii) $H_{\delta_1}\frown H_{\delta_2}=\phi$ for any $\delta_1\in\Delta_{\theta_1}$, $\delta_2\in\Delta_{\theta_2}$ with $\theta_1\neq\theta_2$. Since $\mathfrak{H}_1=\{\smile\{H_\delta;\ \delta\in\Delta_\theta\};\ \theta\in\Theta\}$ belongs to $\{\mathfrak{U}_\lambda;\ \lambda\in\Lambda\}$, we get $\mathfrak{H}_1=\mathfrak{U}_\lambda$ for some $\lambda\in\Lambda$. Therefore there exists an index $\theta_0\in\Theta$ such that $\smile\{H_\delta;\ \delta\in\Delta_{\theta_0}\}=U_{\pi_\lambda(y)}$. Since $f^{-1}(\pi_\lambda^{-1}(\pi_\lambda(y)))=U_{\pi_\lambda(y)}$, $\{H_\delta;\ \delta\in\Delta_{\theta_0}\}$ covers $f^{-1}(y)$. $\{H_\delta\frown f^{-1}(y);\ \delta\in\Delta_{\theta_0}\}$ is evidently a relatively open covering of $f^{-1}(y)$ which consists of countable elements and refines $\mathfrak{G}$. Thus we conclude that $f^{-1}(y)$ has the Lindelöf property and the theorem is completely proved.

References

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