31. Convergence to a Stationary State of the Solution of Some Kind of Differential Equations in a Banach Space

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1. **Introduction.** The purpose of this note is to investigate the behaviour at $t=\infty$ of the solution $x(t)$ of some type of differential equation

$$dx(t)/dt=A(t)x(t)+f(t), \tag{1.1}$$

in a Banach space $\mathfrak{X}$. Roughly speaking, if $A(t)$ and $f(t)$ have some properties and if both of them converge in some sense as $t\to\infty$, then the solution $x(t)$ also converges to some element as $t\to\infty$.

2. **Assumptions and the theorem.** By Σ we denote the set of all the complex numbers λ satisfying $-\theta\leqq\arg\lambda\leqq\theta$, where θ is a fixed angle with $\pi/2<\theta<\pi$.

Assumption 1°. For each t, $0\leqq t<\infty$, $A(t)$ is a closed additive operator which maps a dense subset of $\mathfrak{X}$ into $\mathfrak{X}$. The resolvent set $\rho(A(t))$ of $A(t)$, $0\leqq t<\infty$, contains Σ and the inequality

$$\|(\lambda I-A(t))^{-1}\|\leqq M/(|\lambda|+1) \tag{2.1}$$

is satisfied for each $\lambda\in\Sigma$ and $t\in[0,\infty)$, where M is a positive constant independent of λ and t.

2°. The domain D of $A(t)$ is independent of t and the bounded operator $A(t)A(s)^{-1}$ is Hölder continuous in t in the uniform operator topology for each fixed s;

$$\|A(t)A(s)^{-1}-A(r)A(s)^{-1}\|\leqq K|t-r|^{\rho},$$
$$K>0,\ 0<\rho\leqq 1,\ 0\leqq t,\ r<\infty, \tag{2.2}$$

where K and ρ are positive constants independent of t, r and s.

3°. $f(t)$ is uniformly Hölder continuous in $0\leqq t<\infty$:

$$\|f(t)-f(s)\|\leqq F(t-s)^{\gamma},\ F>0,\ 0<\gamma\leqq 1,\ 0\leqq s,\ t<\infty, \tag{2.3}$$

where F and γ are some constants independent of s and t.

4°. There exist a closed operator $A(\infty)$ with domain D and an element $f(\infty)$ of $\mathfrak{X}$ such that

$$\|(A(t)-A(\infty))A(0)^{-1}\|\to 0,\quad \|f(t)-f(\infty)\|\to 0 \tag{2.4}$$

as $t\to\infty$.

Theorem. Under the assumptions made above, the solution $x(t)$ of (1.1) converges to some element as $t\to\infty$. The limit $x(\infty)$ belongs to D and satisfies

$$A(\infty)x(\infty)+f(\infty)=0. \tag{2.5}$$

Moreover, $dx(t)/dt$ tends to 0 as $t\to\infty$.

It might be possible to make a similar observation about the kind of equations investigated by Prof. T. Kato. Such equations are assumed to satisfy the weaker assumptions that, for some natural number l, $A(t)^{-1/l}$ has a domain independent of t and $A(t)^{1/l}A(s)^{-1/l}$ is Hölder continuous with some exponent $>1-1/l$. But very complicated computations would be needed in order to deduce a similar result as above for such kind of equations.

3. **The proof of the theorem.** By Assumption 1°, each $A(s)$ generates a semi-group $\exp(tA(s))$ of bounded operators and it satisfies

$$\|\exp(tA(s))\| \leqq Ne^{-\alpha t} \tag{3.1}$$

$$\|A(s)\exp(tA(s))\| \leqq Le^{-\alpha t}/t \tag{3.2}$$

for $0<t<\infty$ and $0\leqq s\leqq\infty$, where N, L and α are some positive constants which are dependent only on M and θ. The fundamental solution $U(t,s)$ of (1.1) can be constructed as follows [1]:

$$U(t,s)=\exp((t-s)A(s))+W(t,s), \tag{3.3}$$

$$W(t,s)=\int_s^t \exp((t-\sigma)A(\sigma))R(\sigma,s)d\sigma, \tag{3.4}$$

$$R(t,s)=\sum_{m=1}^{\infty} R_m(t,s), \tag{3.5}$$

$$R_1(t,s)=(A(t)-A(s))\exp((t-s)A(s)), \tag{3.6}$$

$$R_m(t,s)=\int_s^t R_1(t,\sigma)R_{m-1}(\sigma,s)d\sigma, \qquad m=2,3,\cdots. \tag{3.7}$$

For the sake of simplicity, we assume $\rho=1$. In what follows, we denote by C constants which depend only on M, θ, K and $\rho(=1)$. If we put

$$\sup_{\substack{t>s\geqq\tau\\ 0\leqq r\leqq\infty}} \|(A(t)-A(s))A(r)^{-1}\|=\eta(\tau) \tag{3.8}$$

$$\sup_{t>s\geqq\tau} \|f(t)-f(s)\|=\delta(\tau), \tag{3.9}$$

both of the right members tend to 0 as $\tau\to\infty$ by assumptions. By (2.2) and (3.8), we have

$$\|(A(t)-A(s))A(s)^{-1}\| \leqq \sqrt{K}\sqrt{\eta(\tau)}(t-s)^{\frac{1}{2}}, \tag{3.10}$$

hence

$$\|R_1(t,s)\| \leqq \sqrt{K}L\sqrt{\eta(\tau)}(t-s)^{-\frac{1}{2}}e^{-\alpha(t-s)} \tag{3.11}$$

for any $t>s\geqq\tau$. Induction argument shows that for any $m\geqq1$,

$$\|R_m(t,s)\| \leqq(\sqrt{K}L\sqrt{\eta(\tau)})^m e^{-\alpha(t-s)}(t-s)^{\frac{m}{2}-1}\Gamma\left(\frac{1}{2}\right)^m\Big/\Gamma\left(\frac{m}{2}\right). \tag{3.12}$$

Using a rough estimate

$$\sum_{m=1}^{\infty}\alpha^{m-1}/\Gamma(m/2)\leqq 3\exp(2d^2), \quad d>0$$

we obtain

$$\|R(t,s)\| \leqq 3\Gamma(1/2)\sqrt{K}L\sqrt{\eta(\tau)}(t-s)^{-\frac{1}{2}}\exp\{-\beta'(\tau)(t-s)\}, \tag{3.13}$$

where $\beta'(\tau)=\alpha-2\pi KL^2\eta(\tau)$. As in the proof of Lemma 1.2 of [1], we also obtain for $t>\sigma>s\geqq\tau$ that,

$$\|R(t, s)-R(\sigma, s)\|$$
$$\leqq C\sqrt{\eta(\tau)}e^{-\beta(\tau)(\sigma-s)}\left\{\frac{(t-\sigma)^{\frac{1}{2}}}{t-s}+\frac{t-\sigma}{(t-s)(\sigma-s)^{\frac{1}{2}}}\right.$$
$$\left.+\left(\frac{t-\sigma}{t-s}\right)^{\frac{1}{2}}\log\frac{t-s}{t-\sigma}+\left(\frac{t-\sigma}{t-s}\right)^{\frac{1}{2}}\right\} \tag{3.14}$$

where $\beta(\tau)$ is some function less than $\beta'(\tau)$, and that

$$\|A(t)\{\exp((t-s)A(s))-\exp((t-s)A(t))\}\|$$
$$\leqq Ce^{-\alpha(t-s)}\sqrt{\eta(\tau)}(t-s)^{-\frac{1}{2}}. \tag{3.15}$$

The following two inequalities are the direct consequences of (3.14) and (3.15):

$$\left\|\int_s^t A(t)\{\exp((t-\sigma)A(\sigma))-\exp((t-\sigma)A(t))\}R(\sigma, s)d\sigma\right\|$$
$$\leqq C\sqrt{\eta(\tau)}e^{-\beta(\tau)(t-s)}(t-s)^{\frac{1}{2}} \tag{3.16}$$
$$\left\|\int_s^t A(t)\exp((t-\sigma)A(t))(R(\sigma, s)-R(t, s))d\sigma\right\|$$
$$\leqq C\sqrt{\eta(\tau)}e^{-\beta(\tau)(t-s)}\{(t-s)^{-\frac{1}{2}}+1\}. \tag{3.17}$$

By (3.13), (3.16) and (3.17) as well as the formula (1.21) of [1], we readily obtain

$$\|A(t)W(t, s)\|\leqq C\sqrt{\eta(\tau)}e^{-\beta(\tau)(t-s)}\{(t-s)^{\frac{1}{2}}+(t-s)^{-\frac{1}{2}}+1\}. \tag{3.18}$$

On the other hand, by (2.3) and (3.9), we get

$$\|f(t)-f(s)\|\leqq\sqrt{F}\sqrt{\delta(\tau)}(t-s)^{\frac{\gamma}{2}} \tag{3.19}$$

for $t>s\geqq\tau$, therefore we obtain

$$\left\|\int_\tau^t A(t)\exp((t-s)A(s))(f(s)-f(t))ds\right\|$$
$$\leqq\left(\frac{1}{\alpha}+\frac{2}{\gamma}\right)L\sqrt{F}\sqrt{\delta(\tau)} \tag{3.20}$$

assuming $t+1>\tau$ without restriction. Similarly

$$\left\|\int_\tau^t A(t)\{\exp((t-s)A(s))-\exp((t-s)A(t))\}f(s)ds\right\|$$
$$\leqq C\sqrt{\eta(\tau)}\sup_\xi\|f(\xi)\|. \tag{3.21}$$

By (3.1), (3.20) and (3.21) together with a formula in the proof of Theorem 1.3 in [1], we obtain

$$\|A(t)x(t)+f(t)\|\leqq\|A(t)U(t,\tau)x(\tau)\|+Ce^{-\alpha(t-\tau)}\sup\|f(\xi)\|$$
$$+C\sqrt{\eta(\tau)}\sup\|f(\xi)\|+\left(\frac{1}{\alpha}+\frac{2}{\gamma}\right)L\sqrt{F}\sqrt{\delta(\tau)}+C\sqrt{\eta(\tau)}\beta''(\tau)^{-1}\sup\|f(\xi)\|, \tag{3.22}$$

for sufficiently large τ, where $\beta''(\tau)$ is a positive function which is bounded away from 0 for these values of τ. Let ε be any positive number. Then we can select τ so large that the sum of the last three terms of the right member of (3.22) is less than $\varepsilon/2$. After fixing τ arbitrarily as above, we can make the sum of the remaining terms less than $\varepsilon/2$ by taking t sufficiently large. Thus we have proved that

$$A(t)x(t)+f(t)\to 0 \text{ as } t\to\infty. \tag{3.23}$$

As $f(t)$ tends to $f(\infty)$ by assumption, $A(\infty)x(t)=A(\infty)A(t)^{-1}A(t)x(t)$ tends to $-f(\infty)$ and $x(t)=A(\infty)^{-1}A(\infty)x(t)$ to $-A(\infty)^{-1}f(\infty)$ which we denote by $x(\infty)$. Clearly, $x(\infty)$ satisfies (2.5). As $x(t)$ is the solution of (1.1), $dx(t)/dt$ tends to 0 by (3.23).

Reference

[1] H. Tanabe: On the equations of evolution in a Banach space, Osaka Math. Jour., **12**, 363-376 (1960).