29. On the Curvature of Parametric Curves

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1. Introduction. In classical differential geometry the treatment of the curvature of parametric curves is restricted to the case in which the curves are at least twice continuously differentiable. As a contrast to this, on the other hand, we have in real function theory the fundamental theorem of Lebesgue, according to which a function of a real variable is almost everywhere derivable provided it is of bounded variation. And yet the two quantities, curvature and derivative, may be thought to belong by origin to a common mathematical category, in the sense that they both are outcomes of the same process of differentiation, applied once or twice according to the cases. Reflecting upon this fact we are led to surmise that a theory of curvature might be constructed under more general assumptions on the curves than usual. It is the object of the present note to show that such a theory is actually possible. The tools requisite thereto are already obtained in our recent papers [1] to [4].

2. Bend of parametric curves. In what follows the term *interval*, by itself, will always mean a linear interval in its widest sense, i.e. any connected infinite set of real numbers. As usual the prepositive epithets *closed* and *open* for intervals will only be used in connection with finite (that is, bounded) intervals, while we shall term *endless* any interval which is an open set.

Consider a fixed Euclidean space $\boldsymbol{R}^m$ of any dimension $m \geqq 2$. The points of $\boldsymbol{R}^m$ will be regarded as vectors whenever convenient. We shall denote by $p \circ q$ the angle made by any pair of nonvanishing vectors p, q of $\boldsymbol{R}^m$ and contained in the closed interval $[0, \pi]$. By a *parametric curve*, or simply *curve*, in $\boldsymbol{R}^m$ we shall understand an arbitrary mapping of the real line $\boldsymbol{R}$ into the space $\boldsymbol{R}^m$. A curve will be called to be *light*, if it is constant on no intervals.

The letter φ will stand in the present and the next section for a given light curve. We call *bend* of φ on an interval I and denote by $\Omega(\varphi, I)$, the quantity defined as follows. Let Δ be any finite, nonoverlapping sequence of closed intervals $J_1, \cdots, J_n$ $(n \geqq 2)$ situated in I. We require further that none of the increments $\varphi(J_i)$ of φ over them vanish and that these intervals are arranged in Δ in the same order in which they appear in the real line $\boldsymbol{R}$ (so that J_i lies in $\boldsymbol{R}$ on the left of J_{i+1} for $i=1, 2, \cdots, n-1$). Plainly the former requirement is realizable in virtue of the lightness of φ. Now $\Omega(\varphi, I)$ means the supremum, for all such sequences Δ, of the sum of the angles $\varphi(J_i) \diamond\varphi(J_{i+1})$ where i ranges over $1, 2, \cdots, n-1$. When the bend $\Omega(\varphi, I)$ thus defined is finite, the curve φ is termed *straightenable*, or *of bounded bend*, on I. If, further, this is the case for all finite intervals I, we say that φ is *locally straightenable* or *of locally bounded bend*.

As is well known, we have the triangular inequality $x\diamond z+y\diamond z \geqq x\diamond y$ for any triple x, y, z of nonvanishing vectors of the space $\boldsymbol{R}^m$, where the sign of equality holds if especially $x+y=z$. It follows easily from this and the lightness of φ that, in the above definition of bend, we may restrict the sequence $J_1, \cdots, J_n$ of intervals to be such that each neighbouring pair J_i, J_{i+1} taken from them can be enclosed in some closed interval with length less than δ, where δ is a positive number given arbitrarily beforehand.

We developed in our work [1] certain of the fundamental properties of curve bend; but the present note can be read nearly independently of that paper, to which we shall make reference only once later on (§7). We hope to fully utilize the result of [1] in a future paper of ours which will deal with two properties of the curvature of continuous parametric curves.

3. Measure-bend. Henceforward we shall simply write $\Omega(I)$ for $\Omega(\varphi, I)$, since this causes no ambiguity. We go on now to attach to the function Ω an outer measure of Carathéodory, denoted by Ω_* and called *measure-bend* induced by the curve φ. Given any nonvoid set $E\subset \boldsymbol{R}$ we namely define $\Omega_*(E)$ as the infimum of the sum $\Omega(I_1)+\Omega(I_2)+\cdots$, where $\langle I_1, I_2, \cdots\rangle$ is an arbitrary sequence (finite or infinite, but of course countable) of endless intervals which together cover the set E. We define further the value of Ω_* for the void set to be zero. That the set-function Ω_* thus constructed is really an outer Carathéodory measure can be verified without difficulty (cf. Saks [6], p. 43).

We shall prove now the following noteworthy

THEOREM. *We have $\Omega_*(I)=\Omega(I)$ for every endless interval I.*

PROOF. The relation $\Omega_*(I)\leqq\Omega(I)$ being obvious by definition of measure-bend, it is enough to derive the opposite inequality. Clearly $\Omega(I)$ is the supremum of $\Omega(J)$ for closed intervals J in I. Let us fix any such interval J. Consider an arbitrary sequence $I_1, I_2, \cdots$ of endless intervals which together cover I. By *reductio ad absurdum* we easily find the existence of a positive number δ with the property that every closed interval, contained in J and having length less than δ, is contained in one at least of $I_1, I_2, \cdots$. This being so, consider in J any non-overlapping sequence $J_1, \cdots, J_n$ $(n\geqq 2)$ of closed intervals, arranged in the same order in which they appear in $\boldsymbol{R}$, such that none of $\varphi(J_1), \cdots, \varphi(J_n)$ vanish and, moreover, each neighbouring pair J_i, J_{i+1} can be enclosed in some closed interval with length $<\delta$. It follows immediately that the sum of the angles $\varphi(J_i)\diamond\varphi(J_{i+1})$ for $i=1, 2, \cdots, n-1$ cannot exceed the sum $\Omega(I_1)+\Omega(I_2)+\cdots$. Taking the supremum of the former sum we obtain $\Omega(J)\leqq\Omega(I_1)+\Omega(I_2)+\cdots$, and this leads at once to $\Omega(J)\leqq\Omega_*(I)$, the sequence $I_1, I_2, \cdots$ being arbitrary. The interval J has been kept fixed hitherto. We now make it vary arbitrarily and conclude that $\Omega(I)\leqq\Omega_*(I)$, which completes the proof.

REMARK. For another treatment of the matter of this section from a more general standpoint we refer the reader to our forthcoming paper [5].

4. Measure-length. Given a curve ψ in $\boldsymbol{R}^m$ we define as usual the *length* $S(\psi, I)$, or shortly $S(I)$, of ψ over an interval I as the supremum of the sum $|\psi(J_1)|+\cdots+|\psi(J_n)|$, where $\langle J_1, \cdots, J_n\rangle$ is any finite, non-overlapping sequence of closed intervals in I. The meanings of the expressions "rectifiable on I" and "locally rectifiable" are obvious. Further the restriction $s(J)$ of the function $S(I)$ to closed intervals J is plainly an additive interval-function (which is not necessarily finite). We now associate with S a set-function S_* called *measure-length* induced by the curve ψ. The value $S_*(E)$ for any nonvoid set $E\subset\boldsymbol{R}$ is by definition the infimum of the sum $S(I_1)+S(I_2)+\cdots$, where $\langle I_1, I_2, \cdots\rangle$ is an arbitrary sequence (finite or infinite) of endless intervals together covering E; and we define the value of S_* for the void set to be zero. It is then easy to prove the following statement:

THEOREM. *The function S_* thus constructed is an outer measure of Carathéodory which fulfils $S_*(I)=S(I)$ for every endless interval I.*

When in particular the curve ψ is locally rectifiable, the additive interval-function $s(J)$ considered above, being now finite, induces an outer Carathéodory measure s^* by a standard procedure (see Saks [6], p. 64). But it is easily shown that s^* is identical with S_*. Indeed we have $S_*(I)=s^*(I)$ for every endless interval I, since $S_*(I)=S(I)$ by the above theorem and since both $S(I)$ and $s^*(I)$ are the supremum of $S(J)$ for closed intervals J in I. It follows at once that S_* and s^* coincide for open sets. Now, for any set E of real numbers, $s^*(E)$ is equal to the infimum of $s^*(D)$ for open sets D containing E (cf. Saks, p. 68, below), and a corresponding assertion holds also for S_* as immediately seen from the definition of measure-length. Consequently we must have $S_*(E)=s^*(E)$.

REMARK. As already observed in §2 the underlying space $\boldsymbol{R}^m$ is supposed at least 2-dimensional, and this is sufficient for our purpose. Needless to say, however, the considerations of this section remain valid for the 1-dimensional space as well.

5. Spheric measure-length. Let γ be a *spheric* (or, more precisely, *unit-spheric*) curve in $\boldsymbol{R}^m$. This means that $\gamma(t)$ is a unit-vector of $\boldsymbol{R}^m$ for every point t of $\boldsymbol{R}$. For any closed interval $J=[a,b]$, let us denote the angle $\gamma(a)\diamond\gamma(b)$ by $\Gamma(J)$ to shorten our notations. Now the *spheric length* $\Lambda(\gamma,I)$, or simply $\Lambda(I)$, of γ over an interval I is by definition the supremum of the sum $\Gamma(J_1)+\cdots+\Gamma(J_n)$ for finite, non-overlapping sequences $\langle J_1,\cdots,J_n\rangle$ of closed intervals in I. As is readily found, the curve γ is locally rectifiable if, and only if, $\Lambda(I)$ is finite for all finite intervals I. Precisely as S_* was constructed from S in the foregoing §4, we can construct from the spheric length Λ a set-function Λ_* called *spheric measure-length* induced by γ. The spheric analogue of the theorem of §4 reads now as follows:

THEOREM. *The spheric measure-length Λ_* is an outer Carathéodory measure and fulfils the relation $\Lambda_*(I)=\Lambda(I)$ for every endless interval I.*

Finally, if γ is locally rectifiable, the restriction of Λ to the class of all closed intervals is a finite additive interval-function and induces an outer Carathéodory measure which coincides with Λ_*.

6. Open relative derivates of additive set-functions. By an *additive set-function* we shall understand as in our paper [3] any finite set-function defined and additive for the bounded Borel sets in $\boldsymbol{R}$. Given a pair μ,Φ of additive set-functions, suppose μ non-negative. We shall term *open upper μ-derivate* of Φ at a point c of $\boldsymbol{R}$ the upper limit of the ratio $\Phi(I)/\mu(I)$, where I is any open interval which contains c and whose length tends to zero. It should be remarked that for any real number a we mean by the quotient $a/0$ the values $+\infty, 0, -\infty$ according as $a>0, a=0, a<0$ respectively. The *open lower μ-derivate* of Φ at c, we define in a corresponding way. When the two open μ-derivates coincide, we shall call their common value *open μ-derivative* of Φ at the point c.

LEMMA. *The open upper and lower μ-derivates of an additive set-function Φ coincide at each μ-nonlacunar point t with $(\mu)\overline{\Phi}(t)$ and $(\mu)\underline{\Phi}(t)$ respectively.*

PROOF. Let J be a fixed closed interval [or open interval] containing the point t and let I denote an arbitrary open interval containing J [or closed interval contained in J]. Then $\mu(I)$ and $\Phi(I)$ plainly tend respectively to $\mu(J)$ and $\Phi(J)$ as the length $|I|$ tends to $|J|$. Consequently, since $\mu(J)\neq 0$ by hypothesis, the ratio $\Phi(I)/\mu(I)$ tends to $\Phi(J)/\mu(J)$ as $|I|$ tends to $|J|$. The assertion follows now at once.

THEOREM. *Given a nonnegative additive set-function ν besides*

μ, let us write A for the set of all the points none of which are lacunar with respect to both μ and ν. Then,

(i) the relative derivative $(\mu)\nu'(t)$ exists at a point t of A if and only if $(\nu)\mu'(t)$ exists, and when this is the case the two derivatives are mutually reciprocal, i.e. we have

$$(\mu)\nu'(t)=1/(\nu)\mu'(t) \text{ and } (\nu)\mu'(t)=1/(\mu)\nu'(t)$$

under the convention that $1/0=+\infty$ and $1/+\infty=0$. Further, a similar result holds for open relative derivatives as well.

(ii) We have $\tilde{\mu}(\boldsymbol{R}-M)=\tilde{\nu}(\boldsymbol{R}-M)=0$ for the set M of the points t of A at which both $(\mu)\nu'(t)$ and $(\nu)\mu'(t)$ exist.

(iii) The open μ-derivative of ν exists at a point to f A if and only if $(\mu)\nu'(t)$ exists, and when this is the case the two derivatives coincide. Further we may interchange here the roles of μ and ν.

PROOF. Part (i) is evident and part (ii) is an immediate consequence of part (i), Lebesgue's theorem of [3] §4, and the lemma of [3] §5. Finally, part (iii) follows readily from part (i) and the lemma of the present section.

REMARK. As readily seen, the above definitions for open relative derivates and open relative derivative remain meaningful if we consider, instead of the functions μ and Φ, any two set-functions Θ and Ψ respectively, each of which is defined at least for all open intervals K, provided that for every K both $\Theta(K)$ and $\Psi(K)$ are finite and $\Theta(K)$ is nonnegative.

7. Curvature and radius of curvature. Consider a light curve φ which is locally rectifiable and locally straightenable. Let $\Omega(I)$ and $S(I)$ denote respectively the bend and the length of φ over any interval I, in conformity with §2 and §4. By the *curvature* of φ at a point c of $\boldsymbol{R}$ we shall understand the open S-derivative of Ω at c, supposed existent. This will be denoted by $\rho(\varphi, c)$, or more simply, by $\rho(c)$. The *radius of curvature* of φ at c, for which we shall write $\sigma(\varphi, c)$ or $\sigma(c)$, is defined correspondingly by interchanging the functions S and Ω in the above. The consistency of the aforesaid two definitions with the classical ones is a direct consequence of the theorem of [1] §68.

As we have already seen, each of the functions $\Omega(I)$ and $S(I)$ induces an outer Carathéodory measure, Ω_* and S_* respectively, such that $\Omega_*(I)=\Omega(I)$ and $S_*(I)=S(I)$ whenever the interval I is endless. Now both these outer measures assume finite values for bounded sets, since the curve φ is locally rectifiable and locally straightenable. Moreover $S_*(I)>0$ for all intervals I, since $S(I)$ is always positive by lightness of φ. Consequently the theorem of the preceding section, together with the decomposition theorem of [3] §8, readily yields us the following statement which constitutes the main result of the present note:

THEOREM. *For each point t of $\boldsymbol{R}$ the four quantities $\rho(t), \sigma(t)$, $(S_*)\Omega'_*(t)$, $(\Omega_*)S'_*(t)$ satisfy the alternative: either they all exist or else none of them exist. If they exist, $\rho(t)$ and $\sigma(t)$ are mutually reciprocal and we have*

$$\rho(t)=(S_*)\Omega'_*(t) \text{ and } \sigma(t)=(\Omega_*)S'_*(t).$$

Further, the points at which they commonly exist form together a Borel set M such that $S_*(\boldsymbol{R}-M)=\Omega_*(\boldsymbol{R}-M)=0$, and if P and Q denote the Borel sets of the points t of M at which $\rho(t)=+\infty$ and $\sigma(t)=+\infty$ respectively, then for every Borel set $X\subset\boldsymbol{R}$ we have

$$\Omega_*(X)=\Omega_*(PX)+\int_X \rho(t)dS_*(t), \quad S_*(X)=S_*(QX)+\int_X \sigma(t)d\Omega_*(t).$$

REMARK. The additive class of sets which underlies integration in the above, we understand to be of course that of all Borel sets. This accords with the fact that both the functions $\rho(t)$ and $\sigma(t)$ are B-measurable on the set M (cf. the end of [3] §3).

8. A lemma. In this concluding section we state a property of measure-length and spheric measure-length which will be of use on a later occasion. The proof will be given elsewhere.

LEMMA. *Given a locally rectifiable curve ψ in $\boldsymbol{R}^m$, suppose that c is a point of unilateral continuity of ψ. Then we have*

$$S_*(\{c\})=|\psi(c+)-\psi(c-)|$$

for the measure-length S_* induced by ψ. Further, if the curve ψ is especially unit-spheric, we also have the following relation for the spheric measure-length Λ_*:

$$\Lambda_*(\{c\})=\psi(c-)\circ\psi(c+).$$

References

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