28. *On Invariant Groups of m-forms*

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1. The algebraic dimensions of invariant groups (orthogonal groups) of quadratic forms are uniquely determined by the number of their variables. But, those of the invariant groups of m-forms ($m\geq 3$) are not uniquely determined with the number of their variables.

We shall determine two types of invariant groups of m-forms, the one's algebraic dimension is zero[1] and the other's is not zero.

2. Let k be a field of characteristic 0, and V be an n-dimensional vector space over k. We shall say, $F(X)$ is an m-form defined on V, if there exists a symmetric m-linear form $f(X^{(1)}, X^{(2)},\cdots, X^{(m)})$ defined on the direct product of m-copies of V, such that $F(X)=f(X, X,\cdots, X)$. Every homogeneous polynomial with n-variables and of degree m is an m-form.

For an m-form $F(X)$, the F-radical N_F of V, is a subspace of V consisting of all vectors X, which satisfy the equation $f(X^{(1)}, X^{(2)}, \cdots, X^{(m-1)}, X)=0$, for any vectors $X,^{(1)} X^{(2)},\cdots, X^{(m-1)}$, in V.

If $N_F\neq 0$ then we shall say that $F(X)$ is non-degenerate, and if $N_F=0$, degenerate. When $F(X)$ is degenerate, there exists the non-degenerate m-form defined on V/N_F, induced by $F(X)$.

We shall use $E(V)$ to denote the ring of k-linear endomorphisms of V, and $G(F)$, the subset of $E(V)$ consisting of all endomorphisms Λ which leave $F(X)$ invariant, i.e. $F(X)=F(X\cdot\Lambda)$.

Proposition 1. *If $F(X)$ is non-degenerate, $G(F)$ is a group.*

Proof. We have to show that every endomorphism Λ, belonging to $G(F)$ is an automorphism of V.

If Λ is not an automorphism, there exists a non-zero vector X in V, which satisfies $X\cdot\Lambda=0$. Then

$$f(X^{(1)}, X^{(2)},\cdots, X^{(m-1)}, X)=f(X^{(1)}\cdot\Lambda, X^{(2)}\cdot\Lambda,\cdots, X^{(m-1)}\cdot\Lambda, X\cdot\Lambda)=0$$

for any vectors $X^{(1)}, X^{(2)},\cdots, X^{(m-1)}$. This implies that N_F contains non-zero vector X. And this contradiction shows that Λ is an automorphism.

If $F(X)=\sum_{i=1}^{n} a_i x_i^m$, then we shall say that $F(X)$ is a diagonal form.

Proposition 2. *When $F(X)$ is a diagonal form, then $F(X)$ is non-*

1) The algebraic dimensions of the invariant groups of m-forms are zero, if and only if the group is a finite group (cf. C. Chevalley: Théorie des Groupes de Lie, 2, Hermann, Paris (1951)).

degenerate, if and only if $\prod_{i=1}^{n} a_i \neq 0$.

Proof. We can prove easily from the definitions.

Let C_m be the cyclic group generated by a primitive m-th root of 1, and S_n be the symmetric group of n-letters. The multiplicative group of non-zero elements of k will be denoted by k^*. And we shall use $C_m^{(k)}$ to denote $k^* \frown C_m$.

Proposition 3. *Let* $m \geq 3$. *If* k *is algebraically closed and* $F(X)$ *is a non-degenerate diagonal form, then* $G(F)$ *is isomorphic to the semi-direct product of* S_n *and the direct product of* n*-copies of* C_m.

Proof. It follows immediately from Prop. 2 and the condition of Prop. that we can assume all a_i's are 1. And we shall represent an automorphism Λ of V as a non-singular matrix (λ_{ij}). If (λ_{ij}) belongs to $G(F)$, then, comparing the coefficients of the terms $x_\mu \cdot x_\alpha \cdot x_\beta^{m-2}$ (where $\alpha \neq \beta$, $1 \leq \mu \leq n$) of $F(X)$ and $F(X \cdot \Lambda)$, we have the following equations:

$$\sum_{\nu=1}^{n} \lambda_{\mu\nu}(\lambda_{\alpha\nu} \cdot \lambda_{\beta\nu}^{m-2}) = 0$$

for all $\qquad 1 \leq \mu \leq n.$

Because (λ_{ij}) is non-singular, we have

(1) $$\lambda_{\alpha\nu} \cdot \lambda_{\beta\nu} = 0$$

for all $\qquad \alpha \neq \beta, \quad 1 \leq \nu \leq n.$

It is easily seen that all permutation matrices are contained in $G(F)$. So, we can assume $\lambda_{11} \neq 0$, multiplying some permutation matrix to the right side of Λ. Then from (1), $\lambda_{\mu 1} = 0$ for all $2 \leq \mu \leq n$. By the induction with respect to μ, we can find a permutation matrix P and a diagonal matrix D, whose product is equal to Λ. If $D = (d_{ij})$ (where $d_{ij} = 0$, if $i \neq j$), then from the fact that Λ and P belong to $G(F)$, D belongs to $G(F)$. So $d_{ii} = 1$, for all $1 \leq i \leq n$. Thus $G(F)$ is generated by the direct product of n-copies of C_m and S_n.

It is easily seen that the direct product of n-copies of C_m is the normal subgroup of $G(F)$, and the intersection of S_n and the direct product of n-copies of C_m, contains only the identity matrix. This completes the proof.

For non-zero elements a, b, in k, we shall say a and b to be in the same class, if there exists an m-th root of a/b in k. And, if a and b are in the same class, we denote $a \equiv b$. When $F(X)$ is a diagonal form, for the coefficients of $F(X)$ we can assume that

$$\begin{aligned} a_1 \equiv a_2 \equiv \cdots &\equiv a_\alpha \\ a_{\alpha+1} \equiv \cdots &\equiv a_{\alpha+\beta} \\ \cdots\cdots& \\ \cdots &\equiv a_{(\alpha+\beta+\cdots+\gamma)} \\ a_{(\alpha+\beta+\cdots+\gamma)+1} \equiv \cdots &\equiv a_{(\alpha+\beta+\cdots+\gamma+\delta)} \end{aligned}$$

(where $\alpha+\beta+\cdots+\gamma+\delta=n$, and $a_\alpha \neq a_{\alpha+\beta}, \cdots, a_\alpha \equiv a_n, \cdots, a_{(\alpha+\beta+\cdots+\gamma)} \neq a_n$).

THEOREM 1. *Let $m\geq 3$. If $F(X)$ is a non-degenerate diagonal form and its coefficients are as above, then $G(F)$ is isomorphic to the semi-direct product of, the direct product of $S_\alpha, S_\beta, \cdots, S_\gamma$ and S_δ, and the direct product of n-copies of $C_m^{(k)}$.*

Proof. Let $\bar{k}$ be the algebraic closure of k and let $\bar{V}$ be the scalar extension of V with respect to $\bar{k}$. We shall use $\bar{G}(F)$ to denote the invariant group of F, on $\bar{V}$. The general linear groups of V and $\bar{V}$ are denoted by $GL(V)$ and $GL(\bar{V})$, respectively. Then we have $G(F)=\bar{G}(F)\frown GL(V)$.

Let $b_\alpha, b_\beta, \cdots, b_\gamma$ and b_δ, be the m-th root of $a_\alpha, a_\beta, \cdots, a_\gamma$ and a_δ, and let B be the diagonal matrix

$$\begin{pmatrix} b_\alpha & & & & & & 0 \\ & \ddots & & & & & \\ & & b_\alpha & & & & \\ & & & b_\beta & & & \\ & & & & \ddots & & \\ 0 & & & & & & b_\delta \end{pmatrix}.$$

Then, from Prop. 3, we can identify $\bar{G}(F)$ with the group, $B^{-1}\{(C_m\times\cdots\times C_m)\cdot S_n\}B$. So, $G(F)=(C_m\times\cdots\times C_m)\cdot(B^{-1}S_nB)\frown GL(V)$. And from this we can obtain the conclusion.

3. When $F(X)$ is an m-form with m-variables, we shall say $F(X)$ is a multiple form[2] if $F(X)=a\cdot X_1\cdot X_2\cdot \cdots \cdot X_m$ (where $a=0$).

It is easily seen that multiple forms are non-degenerate and we can assume $a=1$.

THEOREM 2. *If $F(X)$ is a multiple form, then $G(F)$ is isomorphic to the semi-direct product of S_m and the direct product of $(m-1)$ copies of k^*.*[3]

Proof. If $\Lambda=(\lambda_{ij})$ belongs to $G(F)$, then, comparing the coefficients of $F(X)$ and $F(X\Lambda)$, we have $\prod_{j=1}^{n}\lambda_{\mu j}=0$ for all $1\leq \mu\leq m$. Multiplying a permutation matrix to the right side of Λ, we can assume $\lambda_{11}\neq 0$ and $\lambda_{12}=0$.

When $\lambda_{11}\neq 0$, $\lambda_{12}=\lambda_{13}=\cdots=\lambda_{1r}=0$, (where $2\leq r<n$), let J_ν be the set of numbers j, for which $\lambda_{j\nu}=0$ $(2\leq\nu\leq r)$. We denote by $e(\alpha)$ the

2) If K is a separable extension of k of degree m, K has a structure of vector space over k. Then, the norm form relative to the extension K/k is an m-form, and it is equivalent to a multiple form in $K\cdot\bar{k}/\bar{k}$, where $\bar{k}$ is an algebraic closure of k (cf. T. Ono: On algebraic groups defined by norm forms of separable extensions, Nagoya Math. J., **11**).

3) In this case the algebraic dimension of $G(F)$ is $(m-1)$.

number of J_ν which contains α $(1\leq\alpha\leq m)$. And, among the α which has the largest $e(\alpha)$, we pick the minimal one and denote it to be α_0.

Let $\{J'_\nu\}$ be all J_ν's that do not contain α_0. And we determine β_0 from these $\{J'_\nu\}$ just as α_0 from $\{J_\nu\}$, and so on. Thus, we have the system $(\alpha_0, \beta_0, \cdots, \gamma_0)$, where $2\leq\alpha_0, \beta_0, \cdots, \gamma_0\leq n$ and $e(\alpha_0)+e(\beta_0)+\cdots+e(\gamma_0)=r-1$. For this system, comparing the coefficients of the term $x_1^{n-r+1}\cdot x_{\alpha_0}^{e(\alpha_0)}\cdot x_{\beta_0}^{e(\beta_0)}\cdot\cdots\cdot x_{\gamma_0}^{e(\gamma_0)}$ of $F(X)$ and $F(X\Lambda)$, we have $\lambda_{1r+1}\cdot\lambda_{1r+2}\cdot\cdots\cdot\lambda_{1n}=0$. So, we can assume $\lambda_{1r+1}=0$, multiplying a permutation matrix to Λ. Thus, we have proved that $\lambda_{12}=\lambda_{13}=\cdots=\lambda_{1n}=0$. Then, from this it is easily seen that $\lambda_{21}=\lambda_{31}=\cdots=\lambda_{n1}=0$. By the induction with respect to m, we can prove that Λ is equal to the product of a diagonal matrix D and a permutation matrix P.

Let $\mathfrak{D}$ be the subgroup of $G(F)$ generated by all diagonal matrices of $G(F)$. Then it is easily seen that $\mathfrak{D}$ is the normal subgroup of $G(F)$ and is isomorphic to the direct product of $(m-1)$ copies of k^*, and $\mathfrak{D}\frown S_m=(I)$. This completes the proof.