51. The Local Structure of an Orbit of a Transformation Group

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A topological group G is said to act on a topological space M when the following conditions are satisfied:

(1) the elements of G are homeomorphisms of M onto itself,

(2) the mapping $(g, x)\to g(x)$ of $G\times M$ onto M is continuous,

(3) $g_1(g_2(x))=(g_1g_2)(x)$ for every $x\in M$ and $g_1, g_2\in G$.

In the following G will denote a locally compact group satisfying the second axiom of countability, G_0 the identity component of G, and M a Hausdorff space throughout this note.

Montgomery and Zippin [7] proved that if G is a compact group acting on a k-dimensional orbit M, then M is locally the topological product of a k-cell by a compact zero dimensional set. In the general case where G is locally compact, as a counter example shows, the above fact is not true, but it holds if only the zero dimensional set is "closed in M" instead of "compact" (the main theorem). As a corollary of this fact it is proved that if G acts transitively and effectively on a finite dimensional connected locally connected space M then G is a Lie group (Corollary 1). Moreover the assumption that M is connected is redundant in this corollary when G/G_0 is compact or G is abelian (Corollary 2).

As G satisfies the second axiom of countability, all factor spaces and orbits of G are separable metric, so that we can make free use of dimension theory (cf. [4]). For topological and group-theoretical terms, we follow the usage of Montgomery and Zippin [6].

The following Lemma 1 was proved by Montgomery and Zippin [7] when G is compact. Using the structure theorem of locally compact groups (cf. [6], p. 175), their proof remains true as it is when G is locally compact and G/G_0 is compact.

Lemma 1 (Montgomery and Zippin [7]). *If G/G_0 is compact and G acts on a finite dimensional orbit $G(x)$, then G is effectively finite dimensional on $G(x)$. In fact, there must be a connected compact invariant subgroup K which is idle on $G(x)$ and such that G/K is finite dimensional.*

Lemma 2. *Let G be a finite dimensional group, and H a closed subgroup of G. Then there is such an arbitrarily small compact local cross section W of cosets of H as the form LZ, where L is a compact local Lie subgroup of G and Z is a compact zero dimensional* *homogeneous set and LZ is homeomorphic to the topological product of L and Z.*

This lemma was essentially proved in [5].

The following lemma is well known and can be proved modifying the arguments in [2]. Therefore we omit the proof (see also [6] and [8]).

Lemma 3 (Gleason [2]). *If L is a local Lie group acting on a completely regular space, such that the action on a given point x is homeomorphic, then there is a closed set C containing x and a neighborhood L' of the identity in L such that the mapping $L'\times C\to L'(C)$ is a homeomorphism onto a neighborhood of x.*

Now we prove the main theorem.

Theorem. *If G acts on a space M, then any finite dimensional orbit $G(x)$ is locally the topological product of a Euclidean cube by a zero dimensional set closed in $G(x)$. If $G(x)$ is locally compact, we can choose this zero dimensional set as a compact one.*

Proof. There exists an open subgroup G' of G such that G'/G_0 is compact. By Lemma 1 there is a compact invariant subgroup K of G' which is idle on $G'(x)$ and such that G'/K is finite dimensional. Let π be the natural mapping of G onto G/K and G_x^* be the image of G'_x under π. Then $G^*=G'/K$ is a finite dimensional group and consequently by Lemma 2 there is a compact local cross section L^*Z^* of cosets of G_x^* which is the topological product of L^* by Z^*, where L^* is a compact local Lie subgroup of G^* and Z^* a zero dimensional compact subset of G^*. Let L be a compact local Lie subgroup of G' which is homeomorphically mapped onto L^* by π (cf. [6], p. 192). Next we choose an element of G' from the complete inverse image of each point in Z^*; from K we choose the identity; and let Z denote the set of the elements of G' thus chosen. Then LZ is a compact local cross section of cosets of G_x in G (cf. [5], p. 343) and LZ is the topological product of L and Z. In particular $\dim L =\dim G/G_x$. Since L acts on x homeomorphically, $\dim L(x)=\dim L$. Moreover $\dim G(x)=\dim G/G_x$ by a theorem of Yamanoshita [9] (see also [5]). On the other hand by Lemma 3, there is a closed set C containing x and a neighborhood L' of the identity in L such that the mapping $L'\times C\to L'(C)$ is a homeomorphism onto a neighborhood of x in $G(x)$. Then it is proved as follows that C is zero dimensional. If C were positive dimensional, there would be a one dimensional subset C' of C by the usual definition of dimension. Since $L'(C')$ is homeomorphic to $L'(x)\times C'$,

$$\dim L'(C')=\dim L'(x)+\dim C' \quad \text{(cf. [3])}.$$

And so $\dim L'(C')>\dim L'(x)=\dim G(x)$. This contradicts the fact that $\dim L'(C')\leqq\dim L'(C)=\dim G(x)$.

Example. Let G be the group of integers acting on a circumference M by rotating integral multiples of an irrational multiple of 2π. Any orbit $G(x)$ is then zero dimensional and locally non-compact.

Corollary 1. *If G acts transitively and effectively on a finite dimensional connected locally connected space M, then G is a Lie group.*

Proof. Since M is a connected manifold, G is a Lie group (cf. [1], p. 106).

Corollary 2. *Let G be a group transitively and effectively acting on a finite dimensional locally connected space M. If G/G_0 is compact or G is abelian, then G is a Lie group.*

Proof. If G/G_0 is compact, G is finite dimensional by Lemma 1. Hence G is a Lie group by a theorem of Bredon [1]. If G is abelian, G_x is equal to the identity for any element x of M. Since the mapping of G onto $G(x)$ defined by $g \to g(x)$ for $g \in G$ is one-to-one open continuous, it is a homeomorphism. Hence G is locally Euclidean, i.e. it is a Lie group.

References

[1] G. E. Bredon: Some theorems on transformation groups, Ann. of Math., **67**, 104–118 (1958).

[2] A. M. Gleason: Spaces with a compact Lie group of transformations, Proc. Amer. Math. Soc., **1**, 35–43 (1950).

[3] W. Hurewicz: Sur la dimension des produits cartésiens, Ann. of Math., **36**, 194–197 (1935).

[4] W. Hurewicz and H. Wallman: Dimension Theory, Princeton Univ. Press (1941).

[5] T. Karube: On the local cross-sections in locally compact groups, J. Math. Soc. Japan, **10**, 343–347 (1958).

[6] D. Montgomery and L. Zippin: Topological Transformation Groups, Interscience Press (1955).

[7] D. Montgomery and L. Zippin: Topological transformation groups I, Ann. of Math., **41**, 778–791 (1940).

[8] P. S. Mostert: Sections in principal fibre spaces, Duke Math. J., **23**, 57–71 (1956).

[9] T. Yamanoshita: On the dimension of homogeneous spaces, J. Math. Soc. Japan, **6**, 151–159 (1954).