60. On the Example of an Inhomogeneous Partial Differential Equation without Distribution Solutions

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1. Let Ω be a domain in Euclidean n-space and let P be a partial differential operator with constant coefficients.

Here we consider the distribution equation

$$PS=F \tag{1}$$

where F is a given distribution in $D'(\Omega)$ and S is the solution in $D'(\Omega)$. It was shown by B. Malgrange [1] that for elliptic operators P and for any domain Ω there is a solution S of (1) and that for hypoelliptic operators P the existence theorem is always valid whenever Ω is a P-convex domain, i.e. to every compact set $K\subset\Omega$ there exists another compact set $K'\subset\Omega$ such that supp $\varphi\subset K'$ for every $\varphi\subset\mathfrak{D}(\Omega)$ such that supp $P'\varphi\subset K$. Furthermore it is easily shown applying usual methods used by several authors that for any geometrically convex domain Ω and for any P the existence theorem is also valid.

In the present note I shall show using a result of F. John's that for the hyperbolic operator the existence theorem is not true for some P-convex domain.

2. To show a counter example we use the following

Lemma 1. Let Ω_i be a bounded subdomain of a domain $\Omega\subset R_n$ such that

$$\Omega\supset\bar{\Omega}_i\not\supset\Omega_{i+1} \qquad \text{for any integer } i=1,2,\cdots,$$

and let K_i be $(n-1)$-dim surfaces such that

$$K_i\subset\Omega_i-\bar{\Omega}_{i-1},$$

$$K_i\to \text{a part of the boundary } \dot{\Omega} \text{ of } \Omega.$$

Furthermore we assume that for some increasing sequence $\{s_i\}$ of integers there exist functions f_i such that

$$\text{supp } f_i\subset\Omega_i \tag{2}$$

$$f_i\in C^{s_i-1}(\Omega_i), \tag{3}$$

$$f_i\in C^{s_i}(U(K_i)-K_i), \tag{4}$$

but for some D^{s_i},

$$D^{s_i}f_i\notin L_{p_i}(U(K_i)) \qquad (\infty>p_i>1), \tag{5}$$

where $U(K_i)$ is an open set, and

$$P'f_i\in C^{s_i-p}(\Omega_i)\frown C^{\infty}(\Omega_i-K) \tag{6}$$

where p is the degree of P and K is a fixed compact subset of Ω.

Then there exists a distribution F such that (1) has no distribution solutions for Ω.

Proof. First of all we show that there exists distribution F_i of order s_i such that supp $F_i \subset$ a small open set $U(K_i)$ and

$$F_i(f_i^* \varphi_\varepsilon) \to \infty \quad \text{as} \quad \varepsilon \to 0, \tag{7}$$

where φ_ε is a C^∞-function such that $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, $\varphi(x)\geqq 0$, supp $\varphi \subset \{x \mid |x|<1\}$ and $\int \varphi(x)dx=1$.

For if we assume that for any $g \in L_{p_i}(U(K_i))$

$$\left| \int_{U(K_i)} g(x)(D^{s_i}(f_i^* \varphi_\varepsilon))(x)dx \right| < \alpha$$

for any ε and for some α. Then $\{D^{s_i}(f_i^* \varphi_\varepsilon) \mid \varepsilon \to 0\}$ is bounded in $L_{p_i}(U(K_i))$. Since L_{p_i} is reflexive, there exists a subsequence of the sequence such that it converges weakly to some ϕ in $L_{p_i}(U(K_i))$. Then we see from (4) that $\phi(x)=D^{s_i}f_i$ for a.e. $x \in U(K_i)-K_i$ and therefore that $D^{s_i}f_i \in L_{p_i}(U(K_i))$ which contradicts (4) and (5).

Setting $F_i=(D'^{s_i})g$ for some $g \in L_{p_i}(U(K_i))$, we see the relation (7).

Let $F=F_1+F_2+\cdots$. Then from the properties of K_i, we see that $F \in \mathfrak{D}'(\Omega)$.

Now we assume that there exists a distribution $S \in \mathfrak{D}'(\Omega)$ such that $PS=F$. Then $S(P'\varphi)=F(\varphi)$ for any $\varphi \in \mathfrak{D}(\Omega)$. Therefore for some neighbourhood N of 0 in $\mathfrak{D}(\Omega)$, if $P'\varphi \in N$, then

$$|F(\varphi)| \leqq 1.$$

On the other hand from (6) it follows that there exist a k and δ_k such that

$$P'(\delta_k f_k^* \varphi_\varepsilon)=\delta_k(P'f_k)^* \varphi_\varepsilon \in N \qquad (\varepsilon \leqq \varepsilon_k).$$

Furthermore from (2) and (3) it follows that for some $\beta>0$,

$$|(F_1+F_2+\cdots+F_{k-1})(\delta_k f_k^* \varphi_\varepsilon)| < \beta \text{ for } \varepsilon \to 0$$

and from (7) that for some ε_k

$$|F_k(\delta_k f_k^* \varphi_{\varepsilon_k})| \geqq 2+\beta.$$

Therefore we see that for some ε_k

$$|F(\delta_k f_k^* \varphi_{\varepsilon_k})| \geqq 2,$$

which is a contradiction.

Lemma 2. Let $P=\dfrac{\partial^2}{\partial t^2}-\dfrac{\partial^2}{\partial x_1^2}-\dfrac{\partial^2}{\partial x_2^2}$ and $x=(x_1, x_2)$. Then for any integer $m>0$ there exists solution g_m such that

$$Pg_m=0 \text{ in } R_3, \tag{8}$$

$$g_m \text{ is analytic for } r=\sqrt{(x_1^2+x_2^2)}<1, \tag{9}$$

$$g_m(t, x) \in C^m \tag{10}$$

$$g_m(t, x) \in C^{m+1} \quad \text{for} \quad r>1, \tag{11}$$

and for some D^{m+1}, $|D^{m+1}g_m(0, x_i)| \to \infty$ for some sequence $x_i(|x_i| \neq 1)$ where x_i converges to a point x which lies in an arbitrary small neighbourhood of $(0, 1, 0)$.

This lemma is proved by F. John [2].

3. Counter example. Let Ω_0 be the cylinder $\subset R^3$ such that (t, x)

$\in \Omega_0$ if and only if $|t|<1$ and $r>3$ and let Ω_1 be the barrel such that $(t, x) \in \Omega_1$ if and only if $r \leqq \varphi(t)$ $(|t| \leqq 1)$ where $\varphi(t)$ is a C^∞-function such that

$$\varphi(t)=2^{-1} \quad \text{for} \quad t=+1, -1$$
$$\varphi(0)=1-\delta \qquad (0<\delta<2^{-1})$$

and $\varphi'(t)>0$ for $t<0$ and $\varphi'(t)<0$ for $t>0$. Let $\Omega=\Omega_0-\Omega_1$, then by the Holmgrem's theorem we see that Ω is a P-convex domain where $P=\dfrac{\partial^2}{\partial t^2}-\Delta_x$.

Now from Lemma 2 we show that there exist subdomains Ω_i, surfaces K_i, numbers s_i and functions f_i such that they satisfy the conditions of Lemma 1. Let Ω'_m be the cylinder $\{(t, x) \mid |t|<3 \cdot 2^{-1}, 1-2^{-(2m+1)}\delta<r<5 \cdot 2^{-1}\}$. Furthermore let $\Omega_m=\Omega'_m-(1-2^{-2m})\,\delta \cdot e$ where $e=(0, 1, 0)$.

Moreover let $p_m(t,x)$ and $q(t, x)$ be C^∞-functions such that $p_m(t, x)=1$ for $r: 1-2^{-(2m+3)}\delta \leqq r \leqq 3 \cdot 2^{-1}$, $p_m(t, x)=0$ for $r: r \leqq 1-2^{-(2m+2)}\delta$ or $r \geqq 2$, $q(t,x)=1$ for $|t| \leqq 3^{-1}$ and $q(t,x)=0$ for $|t|>2^{-1}$. Finally let $g'_m=g_m \cdot p_m \cdot q$ and let $f_m=g'_m(t, x+\delta(1-2^{-2m})e)$.

Then since $\operatorname{supp} g'_m \subset \Omega'_m$, $\operatorname{supp} f_m \subset \Omega_m$ ((2)). From (10) it follows that $f_m \in C^m(\Omega_m)$ ((3)). Let K'_m be a subset of $\{(t, x) \mid |x|=1\}$ which converges to $(0, 1, 0)$ and let $K_m=K'_m-\delta(1-2^{-2m})e$. Then from (9) and (11) it follows that $f_m \in C^{m+1}(U(K_m)-K_m)$ ((4)). Furthermore from (10) and the property of $D^{m+1}g_m$ mentioned above, it implies that $D^{m+1}f_m \notin L^{p_m}(U(K_m))$ for some p_m ((5)). Finally from (8), (9) and (11) we see that $P'f_m \in C^{m+1}(\Omega_m) \frown C^\infty(\Omega_m-K)$, ((6)), where $K=\{(t, x) \mid 3^{-1} \leqq |t| \leqq 2^{-1}, 1-\delta \leqq r \leqq 2+\delta\}\{(t, x) \mid |t| \leqq 3^{-1}, 3 \cdot 2^{-1}-\delta \leqq r \leqq 2+\delta\}$.

4. Remark. Conversely, from the existence theorem for geometrically convex domain and Lemma 1 it follows directly the John-Malgrange theorem [2].

References

[1] B. Malgrange: Séminaire Schwartz, (Equations aux dérivées partial année 1954/55).

[2] F. John: Continuous dependence on data for solutions of partial differential equations with a prescribed bound, Comm. Pure and Apple. Math., **13**, 551-585 (1960).