59. *Heisenberg's Commutation Relation and the Plancherel Theorem*

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1. Let G and X be a locally compact abelian group and its character group, with the Haar measures dg and $d\chi$, respectively. For a Borel subset S of G

(1) $$E(S)f(g)=C_S(g)f(g),$$

where $C_S(g)$ is the characteristic function of S, defines a spectral measure dE acting on $L^2(G)$. It is easy to see that dE satisfies

(2) $$U(g)E(S)=E(gS)U(g),$$

for the regular representation $U(g)(f(\cdot)\rightarrow f(g^{-1}\cdot))$ of G on $L^2(G)$. Using dE, one can define

(3) $$V(\chi)=\int\overline{\chi(g)}dE(g),$$

for each character $\chi\in X$, where the integration ranges over G. It is not hard to see that $V(\chi)$ is a strongly continuous unitary representation of X. The pair $U(g)$ and $V(\chi)$ satisfies the so-called *Heisenberg's commutation relation*:

(4) $$U(g)V(\chi)=\chi(g)V(\chi)U(g).$$

The representations of a pair of unitary groups satisfying (4) are discussed initially by M. H. Stone [4] and J. von Neumann [3] for n-parameter cases. Their Theorem is generalized to locally compact abelian separable groups by G. W. Mackey [2] and improved away the separability by L. H. Loomis [1], which is stated as the following way: *Let $U'(g)$ and $V'(\chi)$ be strongly continuous unitary representations of G and X on a Hilbert space, respectively, satisfying Heisenberg's commutation relation* (4), *then, according to the pair $U'(g)$ and $V'(\chi)$ being irreducible or not, that pair is unitarily equivalent to the pair of the representations $U(g)$ and $V(\chi)$ or to direct sum of their replicas.* This theorem will be referred as Mackey-Loomis' Theorem.

The purpose of the present note is to show that Heisenberg's commutation relation (4), i.e. Mackey-Loomis' Theorem, implies the Plancherel Theorem. Since the proof of Mackey-Loomis does not assume the duality theorem, our task may be observed with some interests.

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2. Let dE' be the spectral measure acting on $L^2(X)$, which is defined by the characteristic functions $C_{S'}$ of Borel sets S' in X. Denote

$$U'(g)=\int \overline{\chi(g)}dE'(\chi).$$

Then $U'(g)$ is a strongly continuous unitary representation of G. It is easy to see that $U'(g)$ is a multiplication operator over $L^2(X)$ by a Borel measurable function $F_g(\chi)$, and hence $U'(gh)=U'(g)U'(h)$ and the strong continuity imply $F_g(\chi)=\chi(g)$, i.e. the inner product $(U'(g)\xi, \eta)=\int \overline{\chi(g)}\xi(\chi)\overline{\eta(\chi)}d\chi$ for every $\xi, \eta \in L^2(X)$. Denote the regular representation of X by $V'(\chi)$. Then the pair $U'(g)$ and $V'(\chi)$ satisfies the commutation relation (4), because $E'(S')V'(\chi)=V'(\chi)E'(\chi^{-1}S')$. Moreover such a pair $U'(g)$ and $V'(\chi)$ is irreducible. Indeed, let a be a bounded operator on $L^2(X)$ commuting with all $U'(g)$ and $V'(\chi)$. Since the von Neumann algebra A generated by $U'(g), g\in G$, is maximally abelian, a belongs to A and a multiplication operator by a Borel measurable function $a(\chi)$. While

$$a(\chi^{-1}\chi_1)\xi(\chi^{-1}\chi_1)=V'(\chi)(a\xi)(\chi_1)=a(V'(\chi)\xi)(\chi_1)=a(\chi_1)\xi(\chi^{-1}\chi_1)$$

implies that $a(\chi^{-1}\chi_1)=a(\chi_1)$ for all χ and a.e. $\chi_1\in X$. Hence a is a constant operator.

Since U' and V' satisfy (4), it is possible to apply Mackey-Loomis' Theorem on U' and V', that is, there exists a unitary transformation T mapping $L^2(G)$ onto $L^2(X)$ such that

(5) $\qquad U'(g)T=TU(g) \qquad$ for every $g\in G$

and

(6) $\qquad V'(\chi)T=TV(\chi) \qquad$ for every $\chi\in X$.

Let F be the transformation mapping $L^1(G)$ into the space of continuous functions on X, defined by the following:

$$(F\varphi)(\chi)=\int \overline{\chi(g)}\varphi(g)dg \qquad \text{for every } \varphi\in L^1(G).$$

Then it will be proved the followings:

LEMMA 1. *$T(\varphi*\psi)=(T\varphi)(F\psi)$ for every $\varphi, \psi\in L^1(G)\frown L^2(G)$, where $\varphi*\psi$ denotes the convolution of φ and ψ.*[1]

Proof. For a.e. $\chi\in X$,

$$T(\varphi*\psi)(\chi)=T\Big[\int\varphi(g^{-1}\cdot)\psi(g)dg\Big](\chi)=T\Big[\int U(g)\varphi(\cdot)\psi(g)dg\Big](\chi),$$

where the integration is L^2-valued Bochner integral and T is bounded, and hence the integration commutes with the operator T, therefore

1) For any bounded linear transformation L from $L^2(G)$ into or onto $L^2(X)$, $U'(g)L=LU(g)$ if and only if $L(\varphi*\psi)=(L\varphi)(F\psi)$. Indeed, the 'only if' part is proved by the same way of the proof of Lemma 1, and the 'if' part may be proved by using of the approximate identity $\{e_\alpha\}$ and by similar method of the proof of Lemma 2. We omit it.

$$=\int\Big[(TU(g)\varphi)(\chi)\psi(g)\Big]dg=\int(U'(g)T\varphi)(\chi)\psi(g)dg$$
$$=\int\overline{\chi(g)}(T\varphi)(\chi)\psi(g)dg=(T\varphi)(\chi)\int\overline{\chi(g)}\psi(g)dg$$
$$=(T\varphi)(F\psi)(\chi).$$

LEMMA 2. *There exists a measurable function* $\alpha(\chi)$ *on* X *such that* $(T\varphi)(\chi)=\alpha(\chi)(F\varphi)(\chi)$, *a.e. in* X, *for every* $\varphi\in L^1(G)\frown L^2(G)$.

Proof. Let $\{e_\alpha\}$ be an approximate identity in $L^1(G)$ generated by a complete neighbourhood system $\{N_\alpha\}$ of the unit of G. Then e_α belongs to $L^1(G)\frown L^2(G)$. For such e_α, $e_\alpha*\varphi\to\varphi$ in both L^1- and L^2-means, therefore, $T(e_\alpha*\varphi)\to T\varphi$ in L^2-mean. While by Lemma 1, $T(e_\alpha*\varphi)=(Te_\alpha)(F\varphi)$. Since $\{F\varphi;\varphi\in L^1(G)\frown L^2(G)\}$ is uniformly dense in $C_\infty(X)$, where $C_\infty(X)$ is the space of all continuous functions on X vanishing at infinite, $lim_\alpha\ Te_\alpha$ exists ($=\alpha$, say) and is clearly measurable. Therefore $(Te_\alpha)(F\varphi)(\chi)\to\alpha(\chi)(F\varphi)(\chi)=(T\varphi)(\chi)$ a.e. in X.

LEMMA 3. *The function* $\alpha(\chi)$ *equals to a non-zero constant* α_0, *say* $|\alpha_0|=1/c$, *a.e. in* X.

Proof. By (6) and Lemma 2, for any $\varphi\in L^1(G)\frown L^2(G)$

$$(TV(\chi)\varphi)(\chi_1)=(V'(\chi)T\varphi)(\chi_1)=V'(\chi)(\alpha\cdot F\varphi)(\chi_1)=\alpha(\chi^{-1}\chi_1)(F\varphi)(\chi^{-1}\chi_1).$$

While, $(TV(\chi)\varphi)(\chi_1)=\alpha(\chi_1)\cdot F(V(\chi)\varphi)(\chi_1)=\alpha(\chi_1)(F\varphi)(\chi^{-1}\chi_1)$. Hence $\alpha(\chi_1)$ $=$constant a.e. in X.

3. Summing up Lemmas 1, 2 and 3, one has immediately

THE PLANCHEREL THEOREM. *Denote the Fourier transformation by* $\mathcal{F}$ *which is defined such that*[2)]

$$\mathcal{F}:\ \varphi\in L^2(G)\ \to\ \frac{1}{c}\int\overline{\chi(g)}\varphi(g)dg.$$

Then, for every $\varphi,\psi\in L^2(G)$

$$\int|(\mathcal{F}\varphi)(\chi)|^2d\chi=\int|\varphi(g)|^2dg \tag{7}$$

and

$$\int(\mathcal{F}\varphi)(\chi)\overline{(\mathcal{F}\psi)(\chi)}d\chi=\int\varphi(g)\overline{\psi(g)}dg. \tag{8}$$

In this Theorem, the formula (8) follows immediately from (7). Let $\mathcal{F}^{-1}$ be the inverse of $\mathcal{F}$, then $\mathcal{F}^{-1}$ is unitary transformation from $L^2(X)$ onto $L^2(G)$, and we prove the following

THE FOURIER INVERSION FORMULA. $(\mathcal{F}^{-1}\xi)(g)=\frac{1}{c}\int\chi(g)\xi(\chi)d\chi$ *a.e.* $g\in G$ *for every* $\xi\in L^1(X)\frown L^2(X)$.

Proof. For every $\psi\in L^1(G)\frown L^2(G)$, the inner product

$$(\mathcal{F}^{-1}\xi,\psi)=(\mathcal{F}\mathcal{F}^{-1}\xi,\mathcal{F}\psi)=(\xi,\mathcal{F}\psi)=\int\xi(\chi)\overline{(\mathcal{F}\psi)(\chi)}d\chi$$

2) For the function $\varphi\in L^2(G)$ not belonging to $L^1(G)$, the transformation is defined by the L^2-approximation.

$$=\frac{1}{c}\int_X\left[\int_G \xi(\chi)\chi(g)\overline{\psi(g)}dg\right]d\chi=\frac{1}{c}\int_G\left[\int_X \xi(\chi)\chi(g)\,d\chi\right]\overline{\psi(g)}dg$$

(by Fubini Theorem)

$$=\left(\frac{1}{c}\int_X \xi(\chi)\chi(\cdot)d\chi,\ \psi\right).$$

Therefore $(\mathcal{F}^{-1}\xi)(g)=\frac{1}{c}\int_X \chi(g)\xi(\chi)d\chi$ a.e. $g\in G$.

Finally, it will be remarked, that $c=\sqrt{2\pi}$ *when G is the real line*, that is,

$$(\mathcal{F}\varphi)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it\lambda}\varphi(t)dt.$$

Indeed, let $C_1(t)$ be the characteristic function of the interval $[-1, 1]$, then by Lemma 2 $(TC_1)(\lambda)=(1/c)(FC_1)(\lambda)=(e^{i\lambda}-e^{-i\lambda})/ic\lambda=(2/c)(\sin\lambda/\lambda)$. Since T is unitary,

$$2=\int_{-\infty}^{\infty}|C_1(t)|^2dt=\int_{-\infty}^{\infty}|(TC_1)(\lambda)|^2d=\left(\frac{2}{c}\right)^2\int_{-\infty}^{\infty}\left|\frac{\sin\lambda}{\lambda}\right|^2d=\frac{4\pi}{c^2},$$

i.e. $c=\sqrt{2\pi}$.

References

[1] L. H. Loomis: Note on a theorem of Mackey, Duke Math. J., **19**, 641-645 (1952).

[2] G. W. Mackey: A theorem of von Neumann and Stone, Duke Math. J., **16**, 313-326 (1949).

[3] J. von Neumann: Eindeutigkeit der Schrödingerchen Operatoren, Math. Ann., **104**, 510-578 (1931).

[4] M. H. Stone: Linear transformations in Hilbert space III, Operational methods and group theory, Proc. Nat. Acad. Sci., **16**, 172-175 (1930).