114. *Note on the Direct Product of Certain Groupoids*[1]

By Takayuki TAMURA, R. B. MERKEL, and J. F. LATIMER

University of California, Davis, U.S.A.

(Comm. by K. SHODA, M.J.A., Oct. 12, 1961)

Consider a semigroup G satisfying

(1.1) There is at least one (left identity) $e \in G$ such that $ea=a$ for all $a \in G$.

(1.2) For any $a \in G$ and for any left identity $e \in G$ there is at least one $b \in G$ such that $ab=e$.

A. H. Clifford [1] and H. B. Mann [2] investigated such systems and they obtained the same result: the system is the direct product of a right singular semigroup and a group. Clifford called such systems multiple groups, Mann called them (l, r) systems, but we call them right groups. In this note we shall define an M-groupoid as generalization of right groups and shall study the conditions for M-groupoids.

DEFINITION. An M-groupoid S is a groupoid[2] (Bruck [4]) which satisfies the following conditions:

(2.1) There is at least one $e \in S$ such that $ex=x$ for all $x \in S$.

(2.2) If y or z is a left identity of S, then $(xy)z=x(yz)$ for all $x \in S$.

(2.3) For any $x \in S$ there is a unique left identity e (which may depend on x) such that $xe=x$.

THEOREM 1. *An M-groupoid S is the direct product of a right singular semigroup and a groupoid with a two-sided identity, and conversely.*

For the proof of this theorem we use the following lemma:

LEMMA. *If and only if a groupoid S has two orthogonal decompositions, it is isomorphic to the direct product of the two factor groupoids obtained from the two decompositions.*

Clifford introduced the notation "orthogonal decomposition" in his paper [1], p. 869, but he did not apply the principle directly. Although this lemma is obvious according to K. Shoda [3], p. 158, we can easily prove it with elementary method.

DEFINITION. A right group S is a groupoid which satisfies the following conditions:

(3.1) For any $x, y, z \in S$, $(xy)z=x(yz)$

(3.2) For any $a, b \in S$, there is a unique $c \in S$ such that $ac=b$.

1) The detail proof will be given elsewhere.

2) A groupoid is a system in which a binary operation is defined.

Using Theorem 1, we have

THEOREM 2. *A right group is isomorphic to the direct product of a right singular semigroup and a group, and conversely.*

Consider various conditions (4.1) through (4.8) and the seven systems, I through VII, of the conditions as follows:

$\exists_1$ means "there is uniquely"

(4.1) $\forall a, b, c \Rightarrow (ab)c = a(bc)$

(4.2) $\forall a, b \Rightarrow \exists_1 c$: $ac = b$

(4.3) $ab = ac \Rightarrow b = c$

(4.4) $\forall a, b \Rightarrow \exists c$: $ac = b$

(4.5) $\exists e$: $\forall a \Rightarrow ea = a$

(4.6) $\forall a, \forall$ (left identity) $e \Rightarrow \exists c$: $ac = e$

(4.7) $\forall a \Rightarrow \exists c$, $\exists$ (left identity) e: $ac = e$

(4.8) $\forall a \Rightarrow \exists c$, $\exists$ (left identity) e: $ca = e$

I : {(4.1), (4.2)}, II : {(4.1), (4.3), (4.4)},

III : {(4.1), (4.4), (4.5)}, IV : {(4.1), (4.5), (4.6)},

V : {(4.1), (4.5), (4.7)}, VI : {(4.1), (4.5), (4.8)},

VII : $S = R \times G$ where R is right singular semigroup and G is a group.

Each of I through VII is characterization of an M-groupoid. In fact we can show

$$\mathrm{II} \rightleftarrows \mathrm{I} \Rightarrow \mathrm{VII} \Rightarrow \mathrm{III} \Rightarrow \mathrm{IV} \Rightarrow \mathrm{V} \Rightarrow \mathrm{VI} \Rightarrow \mathrm{I}.$$

Furthermore consider the following conditions:

(4. 9) If e and f are idempotents, then $ef = f$.

(4.10) S is the set union of some groups.

VIII: {(4.1), (4.9), (4.10)}

THEOREM 3. *A groupoid S is a right group if and only if S satisfies VIII.*

For the proof of this theorem, we may show VIII $\Rightarrow$ VI and VII $\Rightarrow$ VIII.

Adjoin the condition "there is at least one left identity e," to the characterizations I through VI if it is not already included; and replace associativity by the weakened associative law (2.2).

Denote the new systems by I′ through VI′ respectively.

With a counter example, we can show that I′ through VI′ are not necessary conditions for M-groupoids; I′ and II′ are both sufficient conditions, but IV′ through VI′ are not sufficient conditions; while we have no conclusion yet with respect to sufficiency of III′.

Now consider the following modifications VIII$'_1$ of VIII:

VIII$'_1$: {(4.1), (2.2), (4.9), (5.4)}

where (5.4) means that S is the union of disjoint groupoids each of which has a two-sided identity.

Further replace (5.4) by a stronger condition (5.5):

VIII_2': {(4.1), (2.2), (4.9), (5.5)}

where (5.5) says: There is a decomposition $\{S_\alpha\}$[3)] of S such that each S_α is a groupoid with a two-sided identity.

Then we can show that VIII_1' is not a sufficient condition to determine an M-groupoid, while we have

THEOREM 4. *VIII_2' characterizes an M-groupoid.*

Necessity is clear. For the proof of sufficiency we may show that VIII_2' implies (2.3).

References

[1] A. H. Clifford: A system arising from a weakened set of group postulates, Ann. of Math., **34**, 865–871 (1933).

[2] H. B. Mann: On certain systems which are almost groups, Bull. Amer. Math. Soc., **50**, 879–881 (1944).

[3] K. Shoda: General Theory of Algebra (in Jap.), Tokyo, Kyoritsusha (1947).

[4] R. H. Bruck: A Survey of Binary Systems, Berlin, Spring-Verlag (1958).

3) A decomposition means a partition forming a factor groupoid.