110. On the Distribution of the Spectra of Normal Operators in Hilbert Spaces

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We shall define in advance the symbols, which will be used in this paper, as follows:

Definition. Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable and infinite dimensional; let N be a normal operator in $\mathfrak{H}$; let $\rho(N)$, $\sigma(N)=\{z_\nu\}_{\nu=1,2\cdots}$ and $\Delta(N)$ be the resolvent set, the point spectrum and the continuous spectrum of N respectively; let $\{K(z)\}$ be the complex spectral family associated with N; let K_ν be the eigenprojector of N corresponding to the eigenvalue z_ν; and let 0_0 and 0_e be the null operator and the null element in $\mathfrak{H}$ respectively.

We now suppose that λ_0 belongs to $\Delta(N)$ but not to the set of accumulation points of $\sigma(N)$. Then, by applying the factorization of $K(z)$ by the spectral families of the self-adjoint operators $\frac{1}{2}(N+N^*)$ and $\frac{1}{2i}(N-N^*)$ on $\mathfrak{D}(N)$, we can first verify that λ_0 is not an isolated point of $\Delta(N)$. If we next denote by $\Delta_{\varepsilon,\lambda_0}$ the intersection of $\Delta(N)$ and a suitably small ε-neighborhood of λ_0, then, by the application of this result and the fact that $\rho(N)$ is an open set, we can find that the points of $\Delta_{\varepsilon,\lambda_0}$ are continuously distributed. In addition, there is no difficulty in showing that the dimension of $K(\Delta_{\varepsilon,\lambda_0})\mathfrak{H}$ is denumerably infinite, however small $\varepsilon>0$ may be. After these preliminaries, we shall turn to our purpose.

Theorem 1. Let D be a domain in the complex λ-plane whose boundary ∂D is a rectifiable closed Jordan curve. If the closure $\overline{D}$ of D is a subset of the resolvent set $\rho(N)$ of a normal operator N in $\mathfrak{H}$, then

$$(1) \qquad \int_{\partial D}(\lambda I-N)^{-1}d\lambda=0_0,$$

where the curvilinear integration is taken in the counterclockwise direction; and if, conversely, (1) holds, D is a subset of $\rho(N)$.

Proof. We now divide ∂D into n pieces by points $\lambda_1, \lambda_2, \cdots, \lambda_n$ on itself and let $|\lambda_{\alpha+1}-\lambda_\alpha|\rightarrow 0$, $(\alpha=1, 2, \cdots, n; \lambda_{n+1}=\lambda_1)$, by allowing n to become infinite. Then, remembering the facts that $\int_{\partial D}\frac{d\lambda}{\lambda-z}=0$ or $2\pi i$, according as z lies outside or inside ∂D, and that $\rho(N)$ is an open set, we have

$$\begin{aligned}(2)\qquad \int_{\partial D}(\lambda I-N)^{-1}\,d\lambda &=\lim_{n\to\infty}\sum_{\alpha=1}^{n}\int_{G}\frac{1}{\lambda_\alpha-z}dK(z)(\lambda_{\alpha+1}-\lambda_\alpha)\quad (\partial D\subset\rho(N))\\ &=\lim_{n\to\infty}\int_{G}\sum_{\alpha=1}^{n}\Big\{\frac{1}{\lambda_\alpha-z}(\lambda_{\alpha+1}-\lambda_\alpha)\Big\}dK(z)\\ &=\int_{G}\int_{\partial D}\frac{1}{\lambda-z}d\lambda\,dK(z)\\ &=\sum_{\nu}{}'K_\nu\int_{\partial D}\frac{d\lambda}{\lambda-z_\nu}+\int_{\Delta(N)\frown\bar{D}}\int_{\partial D}\frac{1}{\lambda-z}d\lambda\,dK(z)\\ &=2\pi i\sum_{\nu}{}'K_\nu+2\pi i\int_{\Delta(N)\frown\bar{D}}dK(z),\end{aligned}$$

where G and $\sum_{\nu}{}'K_\nu$ denote the complex z-plane and the sum of the eigenprojectors for all eigenvalues $z_\nu\in D$ of N respectively. Since, if we here suppose that $\bar{D}$ belongs to $\rho(N)$, then the first and second terms in the right-hand side of (2) both vanish, we obtain the required equality (1).

Conversely we now suppose that the equality (1) holds. Then, since the domain of the operator $\int_{\partial D}(\lambda I-N)^{-1}\,d\lambda$ is given by $\mathfrak{H}$, we have

$$\begin{aligned}(3)\qquad 0&=\int_{\partial D}((\lambda I-N)^{-1}f,g)d\lambda\quad (f,g\in\mathfrak{H})\\ &=\sum_{\nu}{}'\int_{\partial D}\frac{d\lambda}{\lambda-z_\nu}(K_\nu f,g)+\int_{\Delta(N)\frown\bar{D}}\int_{\partial D}\frac{1}{\lambda-z}d\lambda\,d(K(z)f,g),\end{aligned}$$

as can be found from the method applied to derive (2).

Here we take for f an arbitrary $f_\nu\in K_\nu\mathfrak{H}$ and choose g so that (f_ν,g) never vanishes. Then, since $K(\delta)\neq 0_0$ but $K(\delta)K_\nu=0_0$ for every subset δ of $\Delta(N)$ whose (one-dimensional or two-dimensional) measure is not zero, $K(\delta)f_\nu=0_e$ and hence the second member in the right-hand side of (3) vanishes for $f=f_\nu$. Since moreover $K_\mu K_\nu=0_0$ for every admissible positive integer $\mu\neq\nu$, $K_\mu f_\nu=0_e$. In consequence, we have

$$\int_{\partial D}\frac{d\lambda}{\lambda-z_\nu}=0,$$

which implies that $\bar{D}$ does not contain z_ν. Thus it turns out that $\bar{D}$ never contains $\sigma(N)$ and that by (3)

$$\int_{\Delta(N)\frown\bar{D}}\int_{\partial D}\frac{1}{\lambda-z}d\lambda\,dK(z)=0_0.$$

This equality implies that either $\bar{D}$ does not contain $\Delta(N)$ itself, or else it does not contain all points of $\Delta(N)$ except possibly for its subset of measure zero which consists of those and only those accumulation points of $\sigma(N)$ such that they do not belong to $\sigma(N)$ itself: for otherwise the left member of this equality would be identical with the operator $2\pi i K(\varDelta(N)\frown\overline{D})$ not 0_0, contrary to fact. In addition, even if there exists such a just described subset with zero measure of $\varDelta(N)$ as contained in $\overline{D}$, it lies on ∂D: because $\overline{D}$ does not contain any eigenvalue of N, as we proved before. In consequence, D never contains any point of $\varDelta(N)$.

With these results established above, D is a subset of $\rho(N)$.

Thus the present theorem has been proved.

Theorem 2. Let α be a given complex number, let $D(\alpha)$ be a suitably small domain whose boundary $\partial D(\alpha)$ is a rectifiable closed Jordan curve oriented positively and contains α inside itself, and let $D'(\alpha)$ be an arbitrary domain with the same condition as that for $D(\alpha)$ such that $D'(\alpha)\subset D(\alpha)$ and $\partial D'(\alpha)\frown\partial D(\alpha)=\phi$ where ϕ denotes the empty set. If the integral $\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}d\lambda$ always exists and

$$\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}\,d\lambda=\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda\neq 0_0, \tag{4}$$

then α is an isolated eigenvalue of N.

Proof. Suppose that (4) holds. Then, in the first place, it is evident from Theorem 1 that α does not belong to $\rho(N)$. Next the existence of $\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}d\lambda\neq 0_0$ gives the validity of $K(D'(\alpha))\neq 0_0$, and the relation $\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}d\lambda=\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda$ holding for any small domain $D'(\alpha)$ satisfying the given condition assures that α is not an accumulation point of $\sigma(N)$ in accordance with (2). Hence we suppose, contrary to what we wish to prove, that α is a point of $\varDelta(N)$ such that it does not belong to the set of accumulation points of $\sigma(N)$. Then, since α can not be an isolated point of $\varDelta(N)$ as we pointed out at the beginning of this paper, the inequality $K(\overline{D'(\alpha)})<K(\overline{D(\alpha)})$ would hold for any $D'(\alpha)$ sufficiently smaller than $D(\alpha)$.

On the other hand, as can be found immediately from (2),

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda&=\sum_{\nu}{}'K_\nu+K(\varDelta(N)\frown\overline{D(\alpha)})\\&=\int_{\overline{D(\alpha)}}dK(z)\\&=K(\overline{D(\alpha)}),\end{aligned}$$

and similarly

$$\frac{1}{2\pi i}\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}d\lambda=K(\overline{D'(\alpha)}).$$

In consequence, the just established inequality is in contradiction with (4). Thus α must be an isolated eigenvalue of N, as we wished to prove.

Theorem 3. Let $D(\alpha)$ be the same symbol as in Theorem 2. Then a necessary and sufficient condition that α be an accumulation point of $\sigma(N)$ and $\Delta(N)\frown\overline{D(\alpha)}$ be a set of measure zero (even if there exists a subset of $\Delta(N)$ consisting of accumulation points of $\sigma(N)$ alone in a neighborhood of α) is that the following assertions (a) and (b) hold:

(a) $0_0\neq\frac{1}{2\pi i}\int_{\partial D'(\alpha)}(\lambda I-N)^{-1}\,d\lambda<\frac{1}{2\pi i}\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda<I$, where $D'(\alpha)$ is a sufficiently small domain satisfying the same condition as that for $D(\alpha)$ such that $D'(\alpha)\subset D(\alpha)$;

(b) $\frac{1}{2\pi i}\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda\cdot h=0_e$ for every h belonging to the orthogonal complement $\mathfrak{N}$ of the subspace $\mathfrak{M}$ determined by all eigenelements of N.

Proof. Suppose that (a) and (b) both hold. Then, according to Theorem 1, (a) shows that α never belongs to $\rho(N)$. Since it is at once found by (2) that

$$(5)\qquad \frac{1}{2\pi i}\int_{\partial D(\alpha)}(\lambda I-N)^{-1}d\lambda=\sum_{\nu}{}'K_\nu+K(\Delta(N)\frown\overline{D(\alpha)}),$$

where $\sum_{\nu}{}'K_\nu$ denotes the sum of the eigenprojectors for all eigenvalues z_ν belonging to $D(\alpha)$, we can find with the help of (b) that $K(\Delta(N)\frown\overline{D(\alpha)})h=0_e$ for every $h\in\mathfrak{N}$. On the other hand, since $K(\Delta(N)\frown\overline{D(\alpha)})$ does not exceed the projector of $\mathfrak{H}$ on $\mathfrak{N}$ and hence is orthogonal to the projector of $\mathfrak{H}$ on $\mathfrak{M}$, and since every $f\in\mathfrak{H}$ is expressed in just one way as a sum $g+h$ where $g\in\mathfrak{M}$ and $h\in\mathfrak{N}$, the just established relation shows that $K(\Delta(N)\frown\overline{D(\alpha)})f=0_e$ and hence that $K(\Delta(N)\frown\overline{D(\alpha)})=0_0$. By making use of this result and of the relations (a) and (5), we can conclude that α is an accumulation point of $\sigma(N)$ and that $\Delta(N)\frown\overline{D(\alpha)}$ is a set of measure zero (inclusive of the empty set). Thus the condition is sufficient.

If, conversely, we suppose that α is an accumulation point of $\sigma(N)$ and that $\Delta(N)\frown\overline{D(\alpha)}$ is a set of measure zero (inclusive of the empty set), then we can derive without difficulty (a) and (b) from the equality (5). Hence the condition is necessary.

Thus the proof of the theorem has been finished.

Theorem 4. Let D be a domain whose boundary ∂D is a rectifiable closed Jordan curve, positively oriented, belonging to $\rho(N)$, and let $\mathfrak{N}$ be the same symbol as in Theorem 3. Then a necessary and sufficient condition that D contains $\Delta(N)$ but not any point of $\sigma(N)$ is that the two following relations hold:

$$(\alpha) \qquad \frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-1}d\lambda\cdot f\in\mathfrak{N} \text{ for every } f\in\mathfrak{H};$$

$$(\beta) \qquad \frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-1}d\lambda\cdot h=h \text{ for every } h\in\mathfrak{N}.$$

Proof. Since the integral operator given in (α) or in (β) is a projector, its domain is given by $\mathfrak{H}$.

Now, if (α) holds, we can find at once by (5) that D has no point belonging to $\sigma(N)$; and if, in addition to it, (β) holds, the just obtained result and (5) lead us to the assertion that $K(\Delta(N)\frown\bar{D})$ is the projector of $\mathfrak{H}$ on $\mathfrak{N}$. Hence $\Delta(N)$ must be contained in D by the hypotheses on ∂D. Thus the condition is sufficient.

Conversely we suppose that $\Delta(N)$ is contained in D and that any point of $\sigma(N)$ is not contained in D. Then, since

$$\frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-1}d\lambda=K(\Delta(N))$$

according to (5), it is obvious that (α) and (β) both hold. The condition is therefore necessary.

The proof of the theorem is thus complete.