109. On a Theorem of Levine

By Hisashi CHODA and Katuyosi MATOBA
Osaka Gakugei Daigaku

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1. Following after the notation of Terasaka, let a and i be the closure and interior operations on a topological space E respectively:

1) $A^{aa}=A^{a}$,	1') $A^{ii}=A^{i}$,
2) $(A\smile B)^{a}=A^{a}\smile B^{a}$,	2') $(A\frown B)^{i}=A^{i}\frown B^{i}$,
3) $A\leqq A^{a}$,	3') $A\geqq A^{i}$,
4) $O^{a}=O$,	4') $E^{i}=E$,

where O is the void set. It is well-known that they are related mutually by $i=cac$, where c is the complementation.

Very recently, N. Levine [2] proved the following interesting theorem:

THEOREM 1. *A subset A of E satisfies*

(1) $$A^{ai}=A^{ia},$$

if and only if there are a clopen set H and a nondense set P such that

(2) $$A=(H-P)\smile(P-H):$$

In short, A satisfies (1) *if and only if A is congruent to a clopen set H modulo nondense sets.*

Levine proved the theorem for T_1-spaces. However, the theorem is valid for closure algebras with a few modifications, which will be shown in §2. The remaining part of the proof of the theorem which is contained in §§2–3 is essentially same as that of Levine.

It will be interesting to observe that Levine's theorem has an application which characterizes the Borel sets of a hyperstonean space in terms of the closure and interior operations.

2. The following two identities guarantee that A^{c} *and* A^{a} *satisfy* (1) *whenever A satisfies* (1):

$$A^{cai}=A^{cacac}=A^{aic}=A^{ccacacc}=A^{cia},$$

and

$$A^{aia}=A^{ccacaca}=A^{ciaca}=A^{caica}=A^{cacacca}=A^{caca}=A^{ia}=A^{ai}=A^{aai}.$$

Consequently, A^{i} satisfies (1) if A satisfies (1), since $i=cac$.

It is clear that *a nondense set P and a clopen set H satisfy* (1), since $P^{ai}=O=P^{ia}$ and $H^{ai}=H=H^{ia}$.

It is also true that $H-P$ *satisfies* (1) *for clopen H and nondense P*: If $H>(H-P)^{ia}=(H\frown P^{ci})^{a}=(H\frown P^{ac})^{a}$, then

$$E=H\smile H^{c}>[(H\frown P^{ac})^{a}\smile(H^{c}\frown P^{ac})^{a}]=[(H\smile H^{c})\frown P^{ac}]^{a}=P^{aca}=E$$

shows a contradiction, whence $H=(H-P)^{ia}$. On the other hand,

$$H \geqq (H-P)^{ai} \geqq (H-P)^{iai} = H^i = H.$$

Finally, it needs to show that $(A \smile B)^i = A^i \smile B^i$ *if* $A \leqq H$ *and* $B \leqq H^c$ *for a clopen set* H:

$$H \frown (A \smile B)^i = [H^c \smile (A \smile B)^{ca}]^c = [H \frown (A \smile B)]^{cac}$$
$$= [(H \frown A) \smile (H \frown B)]^i = A^i,$$

and similarly $H^c \frown (A \smile B)^i = B^i$, whence

$$(A \smile B)^i = [H \frown (A \smile B)^i] \smile [H^c \frown (A \smile B)^i] = A^i \smile B^i.$$

3. If A satisfies (2), then $H-P \leqq H$ and $P-H \leqq H^c$ both satisfy (1), whence by the above

$$A^{ia} = [(H-P) \smile (P-H)]^{ia} = (H-P)^{ia} \smile (P-H)^{ia}$$
$$= (H-P)^{ai} \smile (P-H)^{ai} = [(H-P) \smile (P-H)]^{ai} = A^{ai}$$

shows that A satisfies (1).

Conversely, if A satisfies (1), then $H = A^{ia} = A^{ai}$ is clopen. To prove the remainder, it suffices to show that $H-A$ and $A-H$ are both nondense. $(A-H)^{ai} \leqq A^{ai} = H$ and $(A-H)^{ai} \leqq H^{cai} = H^c$ imply $(A-H)^{ai} = O$, whence $A-H$ is nondense. Similarly $(H-A)^{ai} \leqq H$ and $(H-A)^{ai} \leqq A^{cai} \leqq A^{cacac} = A^{iac} = H^c$ imply $(H-A)^{ai} = O$, whence $H-A$ is nondense.

4. A compact Hausdorff space is called *metastonean* provided that it is stonean in the sense of Dixmier [1] and every first category set is nondense. Since it is known by Ogasawara [3] that a compact Hausdorff space is stonean if and only if every Borel set is congruent to a clopen set modulo first category sets, the "if" part of the following theorem is obvious by Levine's theorem:

THEOREM 2. *A compact Hausdorff space is metastonean if and only if any Borel set satisfies* (1).

To prove the converse, it is to be noticed that Ogasawara's theorem cited in the above guarantees the stonean property of the space. Let A be a set of first category, then $A = H \div P$ by the hypothesis, where H is clopen, P is nondense, and $\div$ means the symmetric difference. If H is nonvoid, then $H = A \div P$ is of first category and a contradiction, whence $H = O$, that is, A is nondense. This proves the theorem.

References

[1] J. Dixmier: Sur certains éspaces considérés par M. H. Stone, Summa Brasil. Math., **2**, fasc. 11, (1951).

[2] N. Levine: On commutativity of the closure and interior operators, Amer. Math. Monthly, **68**, 474–477 (1961).

[3] T. Ogasawara: Sokuron II (Lattice theory, in Jap.), Iwanami, Tokyo (1948).