103. On the Fundamental Conjecture of GLC. VI

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The purpose of this note is to show that, a slight modification of the reasoning of our former paper [6] yields a consistency proof of a "restricted analysis" i.e. the theory of natural numbers with the following set concept: $\{x \mid A(x)\}$, where $A(x)$ contains no free set variable, though it may contain bound set variables and free variables on natural numbers. This will follow immediately from the "fundamental conjecture" on "f-closed proof-figures" which latter is to be defined below.

(With regard to this result, reference should be made to G. Kreisel [2], where it is asserted that: If intuitionistic analysis is ω-consistent, then classical analysis is consistent. The proof given in that paper applies in fact only to the "restricted analysis" in the above sense, and in our proof the premise of the ω-consistency of intuitionistic analysis is not needed.)

A formula A in GLC ([3]) is called 'f-closed', if and only if A does not contain any free f-variable. (An f-variable means a variable of higher type where f is an abbreviation of 'formula'. It becomes a formula if we put objects of the correct types to its argument places.) An f-closed formula may contain quantifiers on f-variables.

An inference V left on an f-variable of the form

$$\frac{F(V), \ \Gamma \to \Delta}{V\varphi F(\varphi), \ \Gamma \to \Delta}$$

is called 'f-closed' if and only if the chief formula $V\varphi F(\varphi)$ of the inference is f-closed. (An inference in the proof-figure is called *implicit*, if there exists a cut under the inference, one of whose cut-formulas is a descendant of the chief-formula of the inference. See [4, 2.7], for the precise definition of the notion 'implicit'.)

A proof-figure $\mathfrak{P}$ is called 'f-closed', if and only if the following condition is fulfilled: If $\mathfrak{P}$ contains an implicit V left on an f-variable, then it is f-closed.

Now our "fundamental conjecture on f-closed proof-figures" reads:

THEOREM 1. *The end-sequence of an f-closed proof-figure is provable without cut.*

The logical system consisting of f-closed proof-figure and the theory of natural numbers shall be denoted by f-CNN. We shall prove further the consistency of f-CNN as Theorem 2.

It should be noted that a closed formula
$$\forall x_1 \cdots \forall x_n \exists \varphi \; \forall x(\varphi[x] \;|\!\!-\!\!-\!| \; A(x, x_1, \cdots, x_n))$$
can be proved in *f-CNN* even if *A* may contain many quantifiers on *f*-variables, which means in usual mathematical sense that *f-CNN* contains the set concepts $\{x \mid A(x)\}$ if $A(x)$ does not contain any free set-variable. ($A(x)$ may contain set quantifiers.) Thus *f-CNN* includes the "restricted analysis" above mentioned.

§1. To prove Theorem 1, we have only to perform a slight modification to the proof of Theorem 1 in [6]. In this chapter we shall show how we change that proof for our present purpose.

Let *A* be a formula in an *f*-closed proof-figure. The number of *V*'s on *f*-variable contained in *A* is called the '*f*-degree' of *A*. The words 'regular' and 'isolated degree' in [6] shall be read as '*f*-closed' and '*f*-degree' respectively throughout this paper. We begin our proof by introduction of an inference 'substitution'. Now a proof-figure $\mathfrak{P}$ of order *n* is defined as an *f*-closed proof-figure with 2.5.1 and 2.5.2 in [6] and the following conditions.

1.1. Let *A* be an arbitrary implicit formula in $\mathfrak{P}$. Then the *f*-degree of *A* is less than *n*.

1.2. Let $\mathfrak{J}$ be an arbitrary substitution with the index *i* in $\mathfrak{P}$ and *A* be an arbitrary implicit formula in the upper sequence of $\mathfrak{J}$. If the *f*-degree of *A* is *j*, then $i+j-1 \leq n$.

We define the '*i*-loader' of a sequence in proof-figure of order *n*, and the correspondence of an ordinal diagram of order *n* (cf. [5]) to every sequence of a proof-figure of order *n* in the same way as in [6].

The notions 'a sequence $\mathfrak{S}$ is reducible to sequences $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$' and 'a proof-figure $\mathfrak{P}$ is reduced to proof-figures $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$' are as defined in [6, 3.1].

We have only to show for the proof of our theorem that we can find proof-figures $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ of order *n* for a given proof-figure $\mathfrak{P}$ of order *n* such that '$\mathfrak{P}$ is reduced to $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$'.

We follow 3.2–4.1 of [6]; however we should regard a proof-figure of order *n* considered there as a proof-figure of order *n* just defined. Then we may assume that the end-place of a proof-figure of order *n* contains no logical inference, no implicit beginning sequence and a suitable cut (cf. [4] for 'suitable cut'), moreover, we may assume that every free variable used as an eigenvariable in a proof-figure is different from each other and is not contained in the sequences under the inference in which it is used as an eigenvariable.

Now we define an essential reduction. Let $\mathfrak{P}$ be a proof-figure of order *n* and $\mathfrak{J}$ be a suitable cut in $\mathfrak{P}$. We treat separately several cases according to the form of the outermost logical symbol of the

cut-formulas of $\mathfrak{P}$.

We have only to treat the case that the outermost logical symbol of $\mathfrak{J}$ is V on an f-variable, since other cases can be treated like in [6].

2.1. Let $\mathfrak{P}$ be of the form presented in 4.2.1 of [6].

Let j be the f-degree of $\widetilde{F}(\alpha)$ and i be $n-j+1$. Let $\Gamma_3 \to \Delta_3$ be the i-loader of $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$. Generally $V\varphi F_1(\varphi)$ is different from $V\varphi\widetilde{F}(\varphi)$, as some substitutions may appear between $V\varphi F_1(\varphi), \Pi_1 \to \Lambda_1$ and $V\varphi\widetilde{F}(\varphi), \Pi_2 \to \Lambda_2$. But now this is not the case, because $V\varphi F_1(\varphi)$ contains no free f-variable. Thus $V\varphi F_1(\varphi)$ is $V\varphi\widetilde{F}(\varphi)$.

2.2. Let $\mathfrak{P}'$ be of the form presented in 4.2.3.

Here we should remark that the formula in the upper sequence of $\mathfrak{J}_1$, which is the descendant of $F(\alpha)$, is $\widetilde{F}(\alpha)$. In fact, since $\widetilde{F}(\alpha)$ contains no other free f-variables than α, any substitution does not influence $\widetilde{F}(\alpha)$. So the upper sequence of $\mathfrak{J}_1$ may be denoted as $\Gamma_3 \to \Delta_3, \widetilde{F}(\alpha)$.

2.3. Now we prove that $\mathfrak{P}'$ is a proof-figure of order n. The conditions 2.5. and 2.5.2 in [6] are clear. To prove 1.1 for $\mathfrak{P}'$, it is sufficient to show that the f-degree of $\widetilde{F}(\alpha)$ is less than n. This is clear because $V\varphi\widetilde{F}(\varphi)$ is implicit in $\mathfrak{P}$ and has consequently the f-degree, which is $j+1$, less than n.

The proof of 1.2 for $\mathfrak{P}'$ is given by the first six lines of 4.2.3.2 of [6].

2.4. The proof that σ' is less than σ is given by 4.2.4 of [6].

§2. We obtain the logical system f-CNN of GLC modifying it as follows: 1. Every beginning sequence of f-CNN is of the form $D \to D$ or of the form $a=b, A(a) \to A(b)$ or a "mathematische Grundsequenz" in Gentzen [1]. 2. The following inference-schema called 'induction' is added:

$$\frac{A(a), \Gamma \to \Delta, A(a')}{A(0), \Gamma \to \Delta, A(t)}$$

where a is contained in none of $A(0), \Gamma, \Delta$ and t is an arbitrary term. $A(a)$ and $A(a')$ are called the *chief-formulas* and a is called an *eigenvariable* of this induction. We call every ancestor of $A(a)$ or $A(a')$ *implicit*. 3. The inference V left on an f-variable of the form

$$\frac{F(V), \Gamma \to \Delta}{V\varphi F(\varphi), \Gamma \to \Delta}$$

is restricted by the condition that $V\varphi F(\varphi)$ is f-closed.

Then we have

THEOREM 2. *f-CNN is consistent.*

For the proof of this theorem, read proof of Theorem 2 of [6]

just replacing *RNN* by *f-CNN*.

REMARKS. 1. It should be noted that our present arguments remain correct, if we introduce the following inference-schema to our system:

$$\frac{F(V_1,\cdots, V_n),\ \Gamma \to \Delta}{\forall \varphi_1 \cdots \forall \varphi_n F(\varphi_1,\cdots, \varphi_n),\ \Gamma \to \Delta}$$

where $\forall \varphi_1 \cdots \forall \varphi_n\ F(\varphi_1,\cdots, \varphi_n)$ is *f*-closed. 2. In this paper we use, similarly to [6], ordinal diagrams of order n developed in [5]. By slight modifications we can use the system $O(\{1,\cdots, n\}, I, \phi)$ of ordinal diagrams in [7], where I and ϕ denote the set of all positive integers and the empty set.

References

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