101. On the Existence of Periodic Solutions of Difference-Differential Equations

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In a difference-differential equation

(1) $$x'(t+1)=ax(t+1)+bx(t)+w(t),$$

we suppose that a and b are constant, and $w(t)$ is a continuous and periodic function of the period ω for $-\infty<t<\infty$.

Let $K(t)$ be a kernel function of (1), that is, a solution of (1) under the conditions $K(t)=0$ $(-1\leqq t<0)$, $K(0)=1$, and $w(t)\equiv 0$.

In the sequel, the following condition is always supposed: *every real part of all the roots of the characteristic equation*

$$e^s(s-a)-b=0$$

is less than $-\delta$, *where* δ *is a positive constant.*

Then, $K(t)$ satisfies the equations

$$K'(t+1)=aK(t+1)+bK(t) \qquad (0<t<\infty),$$
$$K'(t)=aK(t) \qquad (0<t<1)$$

and the inequality

$$|K(t)|\leqq ce^{-\delta t} \qquad (0\leqq t<\infty).$$

If we define a function $p(t)$ such that

(2) $$p(t+1)=\int_{-\infty}^{t} w(s)K(t-s)ds,$$

we find that $p(t)$ is a periodic solution of (1) of the period ω, if we formally differentiate (2) and use the periodicity of $w(t)$. This is the fundamental idea in the following discussions.

The purpose of this paper is to discuss the existence of periodic solutions of the equation (1) which has a term $f(t, x, y, \mu)$ or $\mu f(t, x, y)$ instead of $w(t)$. We will establish the following theorems.

THEOREM 1. *In the equation*

(3) $$x'(t+1)=ax(t+1)+bx(t)+f(t, x(t+1), x(t)),$$

where a *and* b *are constant, we suppose that* $f(t, x, y)$ *satisfies the following conditions;*

(i) $f(t, x, y)$ *is continuous for any* t, x, y *and* $f(t, 0, 0)$ *does not identically vanish;*

(ii) $f(t, x, y)$ *is a periodic function of* t *of the period* ω, *where* ω *is a positive constant;*

(iii) $f(t, x, y)$ *satisfies Lipschitz condition such that*

$$|f(t, x_1, y_1)-f(t, x_2, y_2)| \leqq k(|x_1-x_2|+|y_1-y_2|)$$

for any t, x_1, x_2, y_1, y_2, *where* k *is a constant.*

Then, there exists a periodic solution of (1) *of the period* ω, *provided that* $2ck/\delta$ *is less than* 1.

THEOREM 2. *In the equation*

$$x'(t+1)=ax(t+1)+bx(t)+f(t, x(t+1), x(t), \mu), \tag{4}$$

we suppose that $f(t, x, y, \mu)$ *satisfies the following conditions:*

(iv) $f(t, x, y, \mu)$ *is continuous in* (t, x, y, μ) *for any* t, x, y, *and small* $|\mu|$;

(v) $f(t, x, y, \mu)$ *is a periodic function of* t *of the period* ω;

(vi) $f(t, 0, 0, \mu)$ *and* $f(t, 0, 0, 0)$ *do not identically vanish;*

(vii) $f(t, x, y, \mu)$ *satisfies Lipschitz condition such that*

$$|f(t, x_1, y_1, \mu_1)-f(t, x_2, y_2, \mu_2)| \leqq k(|x_1-x_2|+|y_1-y_2|+|\mu_1-\mu_2|)$$

for any t, x_1, x_2, y_1, y_2, *and small* $|\mu_1|, |\mu_2|$, *where* k *is a constant independent on* μ.

Then, there exist periodic solutions $p(t, \mu)$ *of* (4) *and* $p(t)$ *of* (4) *as* $\mu=0$, *provided that* $2ck/\delta$ *is less than* 1. *Furthermore,* $p(t, \mu)$ *uniformly converges to* $p(t)$ *for* $-\infty < t < \infty$ *as* $\mu \to 0$.

THEOREM 3. *In the equation*

$$x'(t+1)=ax(t+1)+bx(t)+\mu f(t, x(t+1), x(t)), \tag{5}$$

we suppose that $f(t, x, y)$ *satisfies the same conditions* (i), (ii), (iii) *as in Theorem* 1.

Then, there exists a periodic solution of (5) *of the period* ω, *provided that* $|\mu| < \delta/2ck$.

Proof of Theorem 1. In order to apply the successive approximation method, we define a sequence $\{x_n(t)\}_0^\infty$ as follows:

$$x_0(t+1)=0,$$

$$x_{n+1}(t+1)=\int_{-\infty}^{t} f(s, x_n(s+1), x_n(s))K(t-s)ds \quad (n=0,1,2,\cdots) \tag{6}$$

for $-\infty < t < \infty$.

Then, it follows that

$$|x_{n+1}(t+1)-x_n(t+1)| \leqq ck\int_{-\infty}^{t}(|x_n(s+1)-x_{n-1}(s+1)| + |x_n(s)-x_{n-1}(s)|)e^{-\delta(t-s)}ds \quad (n=1, 2,\cdots). \tag{7}$$

For $n=0$, we especially have an inequality

$$|x_1(t+1)-x_0(t+1)| \leqq c\int_{-\infty}^{t}|f(s, 0, 0)|e^{-\delta(t-s)}ds. \tag{8}$$

Since $f(t, 0, 0)$ is continuous and periodic for $-\infty < t < \infty$, there exists a constant M such that $|f(t, 0, 0)| \leqq M$ for $-\infty < t < \infty$. Hence, we obtain from (8) that

$$(9)\qquad |x_1(t+1)-x_0(t+1)| \leqq \frac{Mc}{\delta}.$$

Successively applying (7) and (9), we inductively obtain the inequality

$$(10)\qquad |x_{n+1}(t+1)-x_n(t+1)| \leqq \frac{M}{2k}\left(\frac{2ck}{\delta}\right)^{n+1} (n=0, 1, 2, \cdots)$$

for $-\infty < t < \infty$. Hence, the inequality (10) shows us that the sequence $\{x_n(t)\}_0^\infty$ uniformly converges to a function $x(t+1)$ which is a continuous solution of

$$(11)\qquad x(t+1)=\int_{-\infty}^{t} f(s, x(s+1), x(s))K(t-s)ds, \quad (-\infty < t < \infty),$$

provided that $2ck/\delta$ is less than 1.

Now, it is proved that $x(t+1)$ is a periodic solution of the period ω. In fact, we obtain from (6) that

$$x_1(t+1)=\int_{-\infty}^{t} f(s, 0, 0)K(t-s)ds.$$

Then, by using a change of variable and the periodicity of $f(t, 0, 0)$, we obtain

$$\begin{aligned} x_1(t+\omega+1) &= \int_{-\infty}^{t+\omega} f(s, 0, 0)K(t+\omega-s)ds \\ &= \int_{-\infty}^{t} f(s, 0, 0)K(t-s)ds = x_1(t+1), \end{aligned}$$

which means that $x_1(t+1)$ is a periodic function of the period ω. Now, we suppose that every function $x_k(t+1)$ $(k=1, 2, \cdots, n)$ is a periodic function of the period ω. Then, it follows that

$$\begin{aligned} x_{n+1}(t+\omega+1) &= \int_{-\infty}^{t+\omega} f(s, x_n(s+1), x_n(s))K(t+\omega-s)ds \\ &= \int_{-\infty}^{t} f(s+\omega, x_n(s+\omega+1), x_n(s+\omega))K(t-s)ds \\ &= \int_{-\infty}^{t} f(s, x_n(s+1), x_n(s))K(t-s)ds = x_{n+1}(t+1). \end{aligned}$$

Thus, we inductively have the periodicity of all $x_n(t+1)$ $(n=1, 2, \cdots)$, which implies that $x(t+1)$ is a periodic solution of the period ω.

Next, we prove the unicity of $x(t+1)$. Suppose that there exist two solutions $x(t+1)$ and $y(t+1)$ of (11). Then, it follows that

$$|x(t+1)-y(t+1)| \leqq ck\int_{-\infty}^{t} (|x(s+1)-y(s+1)|+|x(s)-y(s)|)e^{-\delta(t-s)}ds.$$

Let $M(t)$ be the maximum of $|x(s)-y(s)|$ over the interval $-\infty$

$< s \leqq t+1$. Then, we find that

$$M(t) \leqq \frac{2ck}{\delta} M(t),$$

which is a contradiction, unless $M(t)$ does identically vanish, for $2ck/\delta$ is less than 1. This proves the uniqueness of solutions of (11).

Finally, we establish that $x(t+1)$ satisfies the equation (3). Differentiating (11) and using the properties of $K(t)$, it follows that

$$\begin{aligned} x'(t+1) &= f(t, x(t+1), x(t)) + \int_{-\infty}^{t} f(s, x(s+1), x(s)) K'(t-s) ds \\ &= f(t, x(t+1), x(t)) + a\int_{-\infty}^{t} fK(t-s)ds + b\int_{-\infty}^{t-1} fK(t-1-s)ds \\ &= ax(t+1) + bx(t) + f(t, x(t+1), x(t)). \end{aligned}$$

This completes the proof.

Proof of Theorem 2. By means of the same method as before, we can establish that there exists a periodic solution of (4), provided that $2ck/\delta$ is less than 1. However, since the perturbed term has a parameter μ, the solution may be dependent on μ. Thus, by $p(t, \mu)$ we denote the solution. For the case $\mu=0$, we already proved the existence of a periodic solution under the same condition, so that we denote it by $p(t)$. From the definition of the successive approximation method, $p(t, \mu)$ and $p(t)$ satisfy the following integral equations respectively:

$$p(t+1, \mu) = \int_{-\infty}^{t} f(s, p(s+1, \mu), p(s, \mu), \mu) K(t-s) ds,$$

$$p(t+1) = \int_{-\infty}^{t} f(s, p(s+1), p(s), 0) K(t-s) ds.$$

Then, it follows that

$$|p(t+1, \mu) - p(t+1)| \leqq ck \int_{-\infty}^{t} (|p(s+1, \mu) - p(s+1)| + |p(s, \mu) - p(s)| + |\mu|) e^{-\delta(t-s)} ds. \tag{12}$$

Denoting by $N(t)$ the maximum of $|p(s, \mu) - p(s)|$ over $-\infty < s \leqq t+1$, (12) leads us to the inequality

$$N(t) \leqq \frac{2ck}{\delta} N(t) + \frac{ck}{\delta} |\mu|.$$

Hence, we have

$$N(t)\left(1 - \frac{2ck}{\delta}\right) \leqq \frac{ck}{\delta} |\mu|. \tag{13}$$

Since $2ck/\delta$ is less than 1, the inequality (13) implies that $N(t)$ tends to zero as $\mu \to 0$ for any t over the interval $-\infty < t < \infty$. This means that $p(t+1, \mu)$ uniformly converges to $p(t+1)$ for $-\infty < t < \infty$ as $\mu \to 0$.

Proof of Theorem 3. We define a sequence $\{x_n(t)\}_0^\infty$ for $-\infty < t < \infty$ as in the proof of Theorem 1 with an exception that we substitute μf for f in (6). Then, by means of the same reason as before, we obtain the inequality

$$|x_{n+1}(t+1) - x_n(t+1)| \leqq \frac{M}{2k}\left(\frac{2|\mu|ck}{\delta}\right)^{n+1} (n=0, 1, 2\cdots),$$

which implies the uniform convergence of $\{x_n(t+1)\}_0^\infty$, provided that $|\mu| < \delta/2ck$. It is apparent that the limiting function $x(t+1)$ is a continuous and periodic solution of (5) of the period ω for $-\infty < t < \infty$.

REMARK. In Theorem 1, by virtue of the inequality (10), we obtain an estimation for the limiting function $x(t+1)$ such that

$$|x(t+1)| \leqq \frac{Mc}{\delta - 2ck}$$

for $-\infty < t < \infty$. Substituting $|\mu|k$ for k in the above inequality, we also obtain an estimation for the limiting function in Theorem 3.