100. On Some Measure-Theoretic Results in Curve Geometry

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1. Preliminary remarks. Let $\boldsymbol{R}^m$ be a Euclidean space of any dimension $m \geqq 1$, where we identify $\boldsymbol{R}^1$ with the real line $\boldsymbol{R}$. By a *curve* (in $\boldsymbol{R}^m$) we shall understand any mapping of $\boldsymbol{R}$ into $\boldsymbol{R}^m$. Thus a curve is no other than a real function when $m=1$. The letter φ will be reserved for a given curve throughout what follows. Let us add that all the sets (and intervals) considered will be situated in $\boldsymbol{R}$ unless stated otherwise or another meaning is obvious from the context. For each set E we define the *length* and the *measure-length* of φ on E as in [1]§37 and in [2]§4 respectively. The former will be denoted by $L(\varphi; E)$ as before, but the latter by $L_*(\varphi; E)$ in this note.

As a matter of fact, the measure-length $L_*(\varphi; E)$ depends not only on the behaviour of φ within the set E but also on its definition for the points outside E, even the case where φ is continuous being no exception to this observation. So long as we are concerned with locally rectifiable curves, however, this does not cause any serious obstacle to the construction of a reasonable theory of measure-length. It is when we step forward beyond such curves that things begin to show themselves unfavourable to us. One way of avoiding the difficulty that thus arises is to abandon the study of the measure-length by itself and to direct our chief interest to certain other set-functions (to be defined in §2) which serve as substitutes for the measure-length and whose values for any set E depend solely on the behaviour of the curve within E. Some of their fundamental properties will constitute the subject matter of the present note.

2. Reduced and Hausdorff measure-lengths of a curve. Given a set E, let us consider an arbitrary sequence (finite or infinite) of its subsets, $\varDelta=\langle E_1, E_2, \cdots\rangle$, such that $[\varDelta]=E_1 \smile E_2 \smile \cdots = E$. The infimum, for all $\varDelta$, of the sum $L(\varphi; \varDelta)=L(\varphi; E_1)+L(\varphi; E_2)+\cdots$ will be termed *reduced measure-length* of the curve φ over the set E and denoted by the symbol $\varXi(\varphi; E)$. Let us now write $\varDelta_\varepsilon$ for $\varDelta$ when especially every E_n has its diameter $\mathrm{d}(E_n)$ smaller than a positive number ε, the diameter of the void set being understood to be zero. Consider the images $\varphi[E_n]$ of the sets E_n under the mapping φ and denote by $\varGamma_\varepsilon(\varphi; E)$ the infimum, for all $\varDelta_\varepsilon$, of the sum $\mathrm{d}(\varphi[E_1])+\mathrm{d}(\varphi[E_2])+\cdots$. When $\varepsilon \to 0$, this infimum plainly tends in a non-decreasing manner to a definite limit, which will be written $\Gamma(\varphi; E)$ and termed *Hausdorff measure-length* of the curve φ over E. We observe that *both $\Xi(\varphi; E)$ and $\Gamma(\varphi; E)$, qua functions of E, are outer Carathéodory measures which vanish for countable sets.* The proof is immediate and left to the reader; in fact, we need only notice that if $\langle I_1, I_2, \cdots \rangle$ is a disjoint sequence of intervals of any type, then the inequality $L(\varphi; E) \geqq L(\varphi; I_1E) + L(\varphi; I_2E) + \cdots$ holds for each set E.

EXAMPLE. Let $f(t)$ be 0 or 1 according as t is rational or irrational. Then both $\Xi(f; E)$ and $\Gamma(f; E)$ clearly vanish identically and are therefore different from $L_*(f; E)$, the latter being $+\infty$ whenever E is nonvoid.

LEMMA. *We have $\Gamma(\varphi; E) \leqq \Xi(\varphi; E) \leqq L_*(\varphi; E)$ for any set E.*

PPOOF. It is obvious that $\Gamma_\varepsilon(\varphi; E) \leqq \Xi(\varphi; E)$ for each $\varepsilon > 0$. Indeed the right-hand side is equal to the infimum of $L(\varphi; \Delta_\varepsilon)$ for all Δ_ε considered above, where we find at once $L(\varphi; \Delta_\varepsilon) \geqq \Gamma_\varepsilon(\varphi; E)$. Since $\Gamma_\varepsilon(\varphi; E) \to \Gamma(\varphi; E)$ as $\varepsilon \to 0$, the inequality $\Gamma(\varphi; E) \leqq \Xi(\varphi; E)$ follows readily.

In order to derive further $\Xi(\varphi; E) \leqq L_*(\varphi; E)$, suppose E nonvoid and consider any sequence $\Theta = \langle I_1, I_2, \cdots \rangle$ of endless intervals covering E. Then, by definition, $L_*(\varphi; E)$ is the infimum of $L(\varphi; \Theta)$ for all such Θ (see [2]§4). On the other hand, writing $\Theta E = \langle I_1E, I_2E, \cdots \rangle$ for short and recalling the definition of $\Xi(\varphi; E)$, we have the evident relation $\Xi(\varphi; E) \leqq L(\varphi; \Theta E) \leqq L(\varphi; \Theta)$. This completes the proof.

3. Continuous functions of locally bounded variation. By a function, by itself, we shall always mean one defined on $\boldsymbol{R}$ and assuming finite real values. With each function F *of locally bounded variation*, that is to say, of bounded variation on closed intervals, we can, by a well-known standard procedure, associate a set-function F^* which is defined for all sets and additive for bounded Borel sets (see Saks [4], p. 64). We require now the following

LEMMA. *If two continuous functions F and G of locally bounded variaton coincide on a bounded set E, we have $F^*(E) = G^*(E)$.*

PROOF. Let us suppose firstly that E is a Borel set. In view of part (iii) of Theorem (6.6) on p. 69 of Saks [4], we may further assume E a closed set. We need only treat the case in which E is non-degenerate, i.e. contains at least two points. Consider the closed interval $I_0 = [\inf E, \sup E]$. Then $F^*(I_0) = F(I_0) = G(I_0) = G^*(I_0)$ by the relation (6.4) on p. 68 of Saks [4] and by hypothesis, and this proves the assertion when $E = I_0$ in particular. Suppose therefore $E \neq I_0$ and let I denote any open interval disjoint from E and with end-points belonging to E. We then find, again using (6.4) just quoted, that $F^*(I) = F(\bar{I}) = G(\bar{I}) = G^*(I)$, the bar indicating the closure operation. On the other hand the additivity of F^* implies that $F^*(I_0) = F^*(E) + \sum F^*(I)$, where the summation extends over all I. The last equality, coupled with its counterpart for G^*, leads at once to $F^*(E) = G^*(E)$ on account of what we have already proved.

We now pass on to the case of general E. According to Theorem (6.7) given on p. 70 of Saks [4] there is a Borel set A subject to the three conditions $E \subset A \subset \bar{E}$, $F^*(A) = F^*(E)$, and $G^*(A) = G^*(E)$. Clearly F and G must then coincide on A. Since A is moreover bounded, the assertion follows readily from what has been established in the above.

4. Locally rectifiable curves. Needless to say, a curve is termed *locally rectifiable* iff (i.e. if and only if) it is rectifiable on all closed intervals.

LEMMA. *If $\xi(t)$ and $\eta(t)$ are a pair of locally rectifiable continuous curves situated in $\boldsymbol{R}^m$, then $L_*(\xi; E) = L_*(\eta; E)$ for any set E on which the two curves coincide.*

PROOF. If we write $s(I) = L(\xi; I)$ for closed intervals I, then $s^*(X) = L_*(\xi; X)$ for every set X as remarked in [2]§4; and a similar result holds of course for the curve η also. In view of this it follows from Theorem (6.7) on p. 70 of Saks [4] that there is a Borel set A fulfilling the conditions $E \subset A \subset \bar{E}$, $L_*(\xi; A) = L_*(\xi; E)$, and $L_*(\eta; A) = L_*(\eta; E)$. Plainly ξ and η then coincide on A. We thus find that E may be assumed a bounded Borel set from now on.

Let $F_i(t)$ and $G_i(t)$ denote the i-th coordinate-functions of ξ and η respectively $(i = 1, 2, \cdots, m)$. We established in our recent paper [3] certain results on the area of interval-surfaces and on the area-measure of set-surfaces. According to the theorem of §57 of that paper, the measure-length $L_*(\xi; E)$ equals the length-measure (*sic!*) of the set-curve $\langle F_1^*, \cdots, F_m^* \rangle$ over E and a similar statement may be made for $L_*(\eta; E)$ as well. But we must have $F_i^*(X) = G_i^*(X)$ for each $i = 1, \cdots, m$ whenever $X \subset E$, on account of the lemma of the foregoing section. So that the two set-curves $\langle F_1^*, \cdots, F_m^* \rangle$ and $\langle G_1^*, \cdots, G_m^* \rangle$ coincide identically within the set E. Collecting the above results, the desired equality $L_*(\xi; E) = L_*(\eta; E)$ follows at once.

THEOREM. *We have $\Xi(\varphi; E) = L_*(\varphi; E) \leqq L(\varphi; E)$ whenever φ is locally rectifiable and continuous at all points of E.*

PROOF. The first half our relation is an immediate consequence of the second half. For let us express E in any manner as the join of a sequence $\Delta = \langle E_1, E_2, \cdots \rangle$ of its subsets. Then, since we may clearly replace E by $E_1, E_2, \cdots$ in the inequality $L_*(\varphi; E) \leqq L(\varphi; E)$, we must have $L_*(\varphi; E) \leqq L_*(\varphi; \Delta) \leqq L(\varphi; \Delta)$. By definition of reduced measure-length this implies that $L_*(\varphi; E) \leqq \Xi(\varphi; E)$, which combined with the lemma of §2 shows that $L_*(\varphi; E) = \Xi(\varphi; E)$.

In order to prove $L_*(\varphi; E) \leqq L(\varphi; E)$, we shall begin with the case in which φ is a continuous curve. Then we may clearly assume E to be a non-degenerate closed set not coinciding with $\boldsymbol{R}$. We now construct from φ another curve ψ as follows. Let us put $\psi(t)=\varphi(t)$ whenever $t \in E$. To define $\psi(t)$ for the remaining points, let I be any connected component of the nonvoid open set $\boldsymbol{R}-E$, so that I is an endless interval. We then extend the definition of $\psi(t)$ to the whole real line so as to make $\psi(t)$ linear on the closure of each interval I. We observe that the definition of ψ on I is unique when I is bounded, since the extremities of I must then belong to E. If, on the other hand, I is unbounded, then ψ is not uniquely determined on I. But this does not matter at all to our purpose.

It is easily seen that the curve ψ, thus constructed, is continuous and locally rectifiable. Since so is also φ by hypothesis, we deduce from the above lemma that $L_*(\varphi; E)=L_*(\psi; E)$. Let now M be the smallest interval containing E, so that $L_*(\psi; E) \leqq L_*(\psi; M)$. We then find without difficulty $L(\psi; E)=L(\psi; M)=L_*(\psi; M)$. On the other hand $L(\varphi; E)=L(\psi; E)$ by construction of ψ. Collecting the above results, we obtain at once $L_*(\varphi; E) \leqq L(\varphi; E)$, Q.E.D.

It remains to establish the same inequality without assuming the continuity of φ. For this purpose let H denote the set of the points of $\boldsymbol{R}$ at which $\varphi(t)$ is discontinuous. Then H is countable since φ is locally rectifiable. In virtue of [1]§94 there is therefore a non-decreasing continuous function $p(u)$ mapping $\boldsymbol{R}$ onto itself and such that the inverse image $p^{-1}(t)$ of a point $t \in \boldsymbol{R}$ under p is non-degenerate and hence a closed interval (called interval of constancy of p), iff t belongs to H. For each point t of $\boldsymbol{R}$ let us now define $q(t)$ to be the point $p^{-1}(t)$ when $t \in \boldsymbol{R}-H$ and to be the middle point of the interval $p^{-1}(t)$ when $t \in H$. Thus defined, $q(t)$ is obviously a strictly increasing function.

This being so, we construct further a curve $\omega(u)$ as follows. For the points $u \in q[\boldsymbol{R}]$, we put simply $\omega(u)=\varphi(p(u))$. To determine $\omega(u)$ for the remaining values of u, let $p^{-1}(t_0)=[a, b]$ be any interval of constancy of p, with c for its middle point. Putting $\omega(a)=\varphi(t_0-)$, $\omega(b)=\varphi(t_0+)$ and noting that $\omega(c)$ is already defined to be $\varphi(t_0)$, we complete the definition of $\omega(u)$ on $[a, b]$ by requiring it to be linear on both $[a, c]$ and $[c, b]$. We then see without difficulty that ω, thus defined on the real line, is a locally rectifiable continuous curve such that $L(\varphi; [t_1, t_2])=L(\omega; [q(t_1), q(t_2)])$ whenever $t_1<t_2$.

The last relation enables us to choose length-functions $s(t)$ and $\sigma(u)$ for the curves $\varphi(t)$ and $\omega(u)$ respectively in such a manner that $s[X]=\sigma[q[X]]$ for every set X. Since $s(t)$ is continuous at all points of E together with $\varphi(t)$ and since $\sigma(u)$ is everywhere continuous together with $\omega(u)$, it follows at once by Theorem (13.3) on p. 100 of Saks [4] that, if we write $N=q[E]$ for the set E of the assertion, then

$$L_*(\varphi; E)=s^*(E)=|s[E]|=|\sigma[N]|=\sigma^*(N)=L_*(\omega; N).$$

On the other hand $\varphi(t)=\omega(q(t))$ for every t by construction of ω, where $q(t)$ is strictly increasing as already mentioned. This gives at once $L(\omega; N)=L(\varphi; E)$. But, since ω is locally rectifiable and continuous, it follows from what we have established in the above that $L_*(\omega; N)\leqq L(\omega; N)$. We thus derive $L_*(\omega; N)\leqq L(\varphi; E)$, which, in combination with $L_*(\varphi; E)=L_*(\omega; N)$ proved already, leads finally to $L_*(\varphi; E)\leqq L(\varphi; E)$. This completes the proof.

5. Intervals on which a curve is continuous. A curve φ will be said *continuous on a set* E iff either E is void, or else E is nonvoid and the subcurve $(\varphi; E)$, i.e. the restriction of φ to E, is continuous (see Saks [4], p. 42, where a similar notion is defined for a function). This must not be confused with the continuity of φ at all points of the set E.

LEMMA. *If φ is continuous on a given interval I (of any type) and if we express I as the join of a sequence $\Delta=\langle E_1, E_2,\cdots\rangle$ of its subsets, then* $\mathrm{d}(\varphi[I])\leqq\mathrm{d}(\varphi[E_1])+\mathrm{d}(\varphi[E_2])+\cdots$.

PROOF. Continuity of φ on I allows us to assume I an endless interval. To simplify our notations, let us write $\Phi(X)=\mathrm{d}(\varphi[X])$ for each set X. Suppose Δ infinite and $\Phi(\Delta)$ finite, as we plainly may, and consider an arbitrary positive number ε. It is easy to choose in I an infinite sequence $\Delta^*=\langle D_1, D_2,\cdots\rangle$ of open sets so as to satisfy the conditions $D_n\supset E_n$ and $\Phi(D_n)<\Phi(E_n)+2^{-n}\varepsilon$ for every $n=1, 2,\cdots$. The last inequality, when summed over all n, gives us $\Phi(\Delta^*)<\Phi(\Delta)+\varepsilon$, and consequently the assertion will be established if we show $\Phi(I)\leqq\Phi(\Delta^*)$. But this is an immediate consequence of the following lemma.

LEMMA. *Suppose that an endless interval I is covered by a family $\mathfrak{M}$ of nonvoid open sets situated in I. By a chain (in $\mathfrak{M}$) we shall understand for the moment any finite sequence $M_1,\cdots, M_n$ consisting of distinct sets ($\mathfrak{M}$) and such that, when $n>1$, every neighbouring pair of elements of the sequence has nonvoid intersection. Such a chain will be said to connect M_1 and M_n (or M_n to M_1). We now assert that each pair of sets ($\mathfrak{M}$) can be connected by a chain.*

PROOF. Let A be any fixed set ($\mathfrak{M}$) and let $\mathfrak{M}_1$ be the subfamily of $\mathfrak{M}$ consisting of all the sets ($\mathfrak{M}$) which can be connected to A by chains, so that $A\in\mathfrak{M}_1$. We shall show that $\mathfrak{M}_1$ coincides with $\mathfrak{M}$. For this purpose, suppose on the contrary that $\mathfrak{M}_2=\mathfrak{M}-\mathfrak{M}_1$ were nonvoid. Then each set ($\mathfrak{M}_2$) must plainly be disjoint from all the sets ($\mathfrak{M}_1$), and so the join of the family $\mathfrak{M}_1$ does not intersect that of $\mathfrak{M}_2$. Accordingly the interval I is partitioned into a pair of non-void open sets. This contradicts the connectedness of I and completes the proof.

THEOREM. *We have $\Gamma(\varphi; I)=\Xi(\varphi; I)=L(\varphi; I)$ for each interval I on which φ is continuous.*

PROOF. Without ambiguity we shall write $\Gamma(X)=\Gamma(\varphi; X)$, etc. for each set X. It follows from the lemma of § 2 and the definition of reduced measure-length that $\Gamma(I)\leqq\Xi(I)\leqq L(I)$. Accordingly we need only ascertain the inequality $L(I)\leqq\Gamma(I)$ in the sequel. For this purpose, let Δ be any disjoint sequence of intervals contained in I. It plainly suffices to derive $\Phi(\Delta)\leqq\Gamma(I)$, where $\Phi(X)$ abbreviates $d(\varphi[X])$ as before for each set X. For each interval J composing Δ, the first lemma of this section gives $\Phi(J)\leqq\Gamma(J)$ by definition of $\Gamma(J)$. Summing the last inequality over all J, we get at once $\Phi(\Delta)\leqq\Gamma(\Delta)\leqq\Gamma(I)$, which completes the proof.

THEOREM. *If I is an interval on which φ is both continuous and rectifiable, then $\Gamma(\Phi; E)=\Xi(\varphi; E)$ for every Borel set $E\subset I$.*

OUTLINED PROOF. Let us denote by $\mathfrak{N}$ the class of all the Borel sets X contained in I and satisfying the equality $\Gamma(\varphi; X)=\Xi(\varphi; X)$. Then every subinterval of I belongs to $\mathfrak{N}$ on account of the foregoing theorem. But it is not difficult to verify that $\mathfrak{N}$ is an additive class. Hence $\mathfrak{N}$ must coincide with the Borel class in I, and the proof is complete.

References

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