130. *Bordered Riemann Surface with Parabolic Double*

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Let W be an open Riemann surface and $\{W_n\}_{n=0}^{\infty}$ be a sequence of regular subregions such that $\overline{W}_n \subset W_{n+1}$ and $W=\bigcup_{n=0}^{\infty} W_n$.[1] Let $u(p)$ be a harmonic function on $W-W_0$. For this u, construct the sequence $\{u_n\}_{n=1}^{\infty}$ of functions $u_n(p)$ continuous on $W-W_0$ and harmonic on $W_n-\overline{W}_0$ such that $u_n=u$ on ∂W_0 and $u_n=c$ on $W-W_n$, where c is a fixed constant. Assume

(1) $$\lim_n u_n(p)=u(p)$$

uniformly on each compact subset of $W-W_0$. For brevity, we denote this fact by $u=c$ *on the ideal boundary* ∂W *of* W. By using Dirichlet principle, it is easily seen that the Dirichlet integral $D_{W-\overline{W}_0}(u)$ is finite and

(2) $$\lim_n D_{W-\overline{W}_0}(u-u_n)=0.$$

It is also clear that

(3) $$|u_n(p)|,\ |u(p)|\leq \max(\max_{\partial W_0}|u(p)|,\ |c|)$$

on $W-W_0$. The Green function $g(p,q)$ with pole q in W_0 or the harmonic measure $w(p;W-W_0,\partial W)$ of ∂W is an important example of such a function u, i.e. $g(p,q)=0$ and $w(p;W-W_0,\partial W)=1$ on ∂W respectively. We put

$$m=\min_{\partial W_0} u(p) \text{ and } M=\max_{\partial W_0} u(p)$$

and assume that $c<m$ (or $c>M$). Choose an arbitrary number t such that

$$c<t<m \text{ (or } c>t>M)$$

and let R be a component of the open set $\{p\in W-\overline{W}_0;\ u(p)>t\}\cup\overline{W}_0$ (or $\{p\in W-\overline{W}_0;\ u(p)<t\}\cup\overline{W}_0$). It is easy to see that $R=W$ if and only if W is parabolic. Hence from now on we assume that W is hyperbolic. Then R is a bordered Riemann surface with border $\Gamma=\{p\in W;\ u(p)=t,\ du(p)\neq 0\}\cap\overline{R}$. Each component of the closure $\overline{\Gamma}$ of Γ is a piecewise analytic curve in W. Construct the double $\hat{R}$ of R along Γ. Z. Kuramochi pointed out the following fact:[2]

THEOREM. *The surface* $\hat{R}$ *is closed or parabolic.*

The proof of this theorem given by Kuramochi is based on his

1) For terminologies and notions not explained in this note, refer to Ahlfors-Sario's book, Riemann surfaces, Princeton, 1960.

2) Proc. Japan Acad., **32**, 25-30 (1955).

theory of capacities on the ideal boundary of a Riemann surface. Recently, Y. Kusunoki and S. Mori have given in their joint work[3] an elegant alternating proof of this theorem using the theory of Royden's compactification of a Riemann surface. But in view of the fundamental feature of this theorem, it is of some interest to give an elementary simple direct proof, and it is the aim of this note.

Proof of Theorem. We shall prove the theorem under the assumption $c<t<m$ and R is not closed. In the case $c>t>M$, we have only to replace u by $-u$. By considering $u-c$ instead of u, we may also assume $c=0$ so that

$$0<t<m.$$

Let $w(p)=w(p;\hat{R}-\hat{K},\partial\hat{R})$ be the harmonic measure of the ideal boundary $\partial\hat{R}$ of $\hat{R}$, i.e. $w=0$ on $\partial\hat{K}$ and $w=1$ on $\partial\hat{R}$, where $\hat{K}$ consists of $K=\overline{W}_0\frown R$ and its symmetric image in $\hat{R}$. Clearly the Dirichlet integral of w over $\hat{R}-\hat{K}$ and hence over $R-K$ is finite. Now we consider $w(p)$ on $R-K$. Then clearly we have

$$\frac{\partial w}{\partial \nu}ds=0 \tag{4}$$

on Γ, where $\partial/\partial\nu$ and ds denote the inner normal derivative with respect to R and the line element of Γ respectively. We define the sequence $\{w_n\}_{n=1}$ of functions on $R-K$ by

$$w_n(p)=\frac{w(p)}{u(p)}u_n(p).$$

Then clearly

$$w_n(p)=0 \qquad \text{for } p\in(\partial W_n\frown R)\smile\partial K. \tag{5}$$

Moreover we have

$$\lim_n D_{R-K}(w_n-w)=0. \tag{6}$$

In fact, putting $f(p)=w(p)/u(p)$ on $R-K$, we see that

$$\|f\|_{R-K}\leq 1/t<\infty, \tag{7}$$

where $\|f\|_{R-K}$ denotes the supremum of $|f(p)|$ on $R-K$. By Schwarz's inequality,

$$D_{R-K}(f)\leq(\sqrt{D_{R-K}(w)}/t+\sqrt{D_{R-K}(u)}/t^2)^2<\infty. \tag{8}$$

Now take an arbitrary compact subset F in $R-K$. Again by using Schwarz's inequality, the relation $w_n-w=(u_n-u)f$ and (3), we see that

$$D_F(w_n-w)\leq(\|u-u_n\|_F\sqrt{D_F(f)}+\|f\|_F\sqrt{D_F(u-u_n)})^2$$

and

$$D_{R-K-F}(w_n-w)\leq(2\|u\|_{R-K}\sqrt{D_{R-K-F}(f)}+\|f\|_{R-K}\sqrt{D_{R-K}(u-u_n)})^2.$$

Hence by applying (1) and (2) to the above,

$$\overline{\lim}_n D_{R-K}(w_n-w)\leq 4\|u\|^2_{R-K}D_{R-K-F}(f).$$

3) Japanese J. Math., **24**, 52-56 (1960).

In view of (8), we can make the right hand term arbitrarily small by taking F large enough in $R-K$. Thus we have (6). From (6)

$$D_{R-K}(w)=\lim_n D_{R-K}(w_n, w). \tag{9}$$

On the other hand, by Green's formula and by noticing that w is harmonic on $R-K$, we get

$$\begin{aligned} D_{R-K}(w_n, w) &= D_{R\frown W_n-K}(w_n, w) \\ &= \int_{\Gamma\frown W_n} w_n \frac{\partial w}{\partial \nu} ds + \int_{(\partial W_n \frown R)\smile \partial K} w_n \frac{\partial w}{\partial \nu} ds. \end{aligned}$$

In view of (4) (resp. (5)), the first (resp. second) term of the right hand side of the above is zero. Thus $D_{R-K}(w_n, w)=0$. Combining this with (9), we get $D_{R-K}(w)=0$. As $w=0$ on ∂K, we obtain

$$w(p;\, \hat{R}-\hat{K},\, \partial\hat{R})\equiv 0$$

on $R-K$ and hence on $\hat{R}-\hat{K}$. This shows that R is parabolic.