124. Open Basis and Continuous Mappings

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Let X and Y be topological spaces and let $f(X)=Y$ be an open continuous mapping. B. Ponomarev [6] has recently obtained the following theorem: *if X has a point-countable open base and the inverse image $f^{-1}(y)$ is separable for each point y of Y, then Y has a point-countable open base.* It is interesting to know under what conditions the property of the open base of X will be preserved under the open (or closed) continuous mapping.

In this note, we shall deal with this problem.

1. Open basis and closed mappings. At the beginning of this section, we shall recall a definition. Let $\mathfrak{U}=\{U_\alpha\}$ be a open base of X. If $\mathfrak{U}$ is star-countable (or locally countable or point-countable), we say that $\mathfrak{U}$ is a star-countable (or locally countable or point-countable) open base.

Theorem 1. *If f is a closed continuous mapping from a topological space X with a star-countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is connected and countably compact for each point y of Y, then Y has also a star-countable open base.*

Proof. Let $\mathfrak{U}=\{U_\alpha\}$ be a star-countable open base of X, then, by K. Morita's theorem [3] (or Yu. Smirnov's lemma [8]), we can see that X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma}A_\gamma$, $A_\gamma\frown A_{\gamma'}=\phi$, $\gamma\neq\gamma'$, $\gamma,\gamma'\in\Gamma$, and $A_\gamma=\smile\{U_\alpha\in\mathfrak{U}_\gamma\}$ where $\mathfrak{U}_\gamma$ is a countable subfamily of $\mathfrak{U}$. Since $f^{-1}(y)$ is connected, we have $f^{-1}f(A_\gamma)=A_\gamma$ for each γ of Γ. Since f is a closed continuous mapping, $f(A_\gamma)$ is open. And moreover $f(A_\gamma)\frown f(A_{\gamma'})=\phi$ for $\gamma\neq\gamma'$. Since each A_γ is perfectly separable and $f^{-1}(y)$ is countably compact, $f^{-1}(y)$ is compact. Hence each $f(A_\gamma)$ is perfectly separable because f is closed and continuous. Therefore Y has a star-countable open base. This completes the proof.

Theorem 2. *Let f be a closed continuous mapping from a topological space X with a star-countable open base onto a topological space Y such that the point inverse image $f^{-1}(y)$ is compact (or separable and countably compact) for each point y of Y. Then Y has a star-countable open base if and only if any open covering of Y has a star-countable open refininement.*

Proof. As the "only if" part is obvious, we shall prove the "if" part. Let $\mathfrak{U}=\{U_\alpha\}$ be a star-countable open base of X. Since $f^{-1}(y)$

is compact for each point y of Y, $\mathfrak{U}_y=\{U_\alpha \mid f^{-1}(y)\frown U_\alpha \neq \phi,\ U_\alpha \in \mathfrak{U}\}$ is countable and $\mathfrak{U}_y$ has a finite subcovering of $f^{-1}(y)$. The collection of all finite subcoverings of $f^{-1}(y)$ is countable. Let $w_k^y(k=1,2,\cdots)$ denote this collection and let $H_k^y=\smile\{U_\alpha \mid U_\alpha \in w_k^y\}$. Then $y\in f((H_k^y)_0)$ $=V_k(y)$ where $(H_k^y)_0=\smile\{f^{-1}(y')\mid f^{-1}(y')\subset H_k^y\}$. Then $f^{-1}(V_k(y))=(H_k^y)_0$ and $\mathfrak{H}=\{V_k(y)\mid k=1,2,\cdots; y\in Y\}$ is an open covering of Y because f is a closed continuous mapping. By assumption, $\mathfrak{H}$ has a star-countable open refinement $\mathfrak{R}=\{R_\beta \mid \beta\in B\}$. Then, for each R_β of $\mathfrak{R}$, there exists a set H_k^y such that $f^{-1}(R_\beta)\subset H_k^y$. Hence $\{U_\alpha \mid f^{-1}(R_\beta)\frown U_\alpha \neq \phi,$ $U_\alpha \in \mathfrak{U}\}$ is countable. We denote this collection of sets by $\{U_{\alpha_i} \mid i$ $=1,2,\cdots\}$ and let $V_{\alpha_i}^\beta=f^{-1}(R_\beta)\frown U_{\alpha_i}\,(i=1,2,\cdots)$. Let $\{S_k^\beta \mid k=1,2,\cdots\}$ denote all finite subfamlies of $\{V_{\alpha_i}^\beta \mid i=1,2,\cdots\}$ and let $K_k^\beta=\smile\{V_{\alpha_j}^\beta \mid$ $V_{\alpha_j}^\beta \in S_k^\beta\}$ and let $W_k^\beta=f((K_k^\beta)_0)$. Then $\mathfrak{W}=\{W_k^\beta \mid k=1,2,\cdots; \beta\in B\}$ is an open covering of Y. We shall next prove that $\mathfrak{W}$ is the star-countable open base. Let y be any point of any given open set G, then there exists a set R_β of $\mathfrak{R}$ such that $y\in R_\beta$. Then $f^{-1}(y)\subset f^{-1}(G)$ and $f^{-1}(y)\subset f^{-1}(R_\beta)$. Since $\mathfrak{U}$ is an open base, for any point x of $f^{-1}(y)$, there exists a set U_{α_x} of $\mathfrak{U}$ such that $x\in U_{\alpha_x}\subset f^{-1}(G)$. Since $f^{-1}(y)$ is compact, there exists a set H_k^y such that $f^{-1}(y)\subset H_k^y\subset f^{-1}(G)$. Hence $f^{-1}(y)\subset H_k^y\frown f^{-1}(R_\beta)\subset f^{-1}(G)$. Then there exists a set W_j^β such that $y\in W_j^\beta\subset G$. Therefore $\mathfrak{W}$ is the open base of Y. Since $\mathfrak{R}$ is star-countable and $f^{-1}(R_\beta)\supset\bigcup_{k=1}^{\infty} f^{-1}(W_k^\beta)$, $\mathfrak{W}$ is star-countable. Therefore Y has a star-countable open base.

We shall next prove the case when $f^{-1}(y)$ is separable and countably compact for each point y of Y. We need only prove the "if" part. Since $f^{-1}(y)$ is separable, $\mathfrak{U}_y=\{U_\alpha \mid U_\alpha\frown f^{-1}(y)\neq\phi,\ U_\alpha\in\mathfrak{U}\}$ is countable [6, Lemma 2]. Hence, from the countable compactness of $f^{-1}(y)$, $\mathfrak{U}_y$ has a finite subcovering of $f^{-1}(y)$. Then, as the remainder of the proof can be carried out in the same way as the case when $f^{-1}(y)$ is compact, we omit the proof. This completes the proof.

We can prove the following theorem in the similar argument as the proof of Theorem 2.

Theorem 3. *Let f be a closed continuous mapping from a topological space X with a locally countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is compact (or separable and countably compact) for each point y of Y. Then Y has a locally countable open base if and only if any open covering of Y has a locally countable open refinement.*

Theorem 4. *Let f be a closed continuous mapping from a paracompact topological space X with a locally countable open base onto a regular topological space Y such that the inverse image $f^{-1}(y)$ is compact for each point y of Y, then Y has a locally countable* *open base.*

Proof. From the proof of the theorem due to K. Morita and the author [4, Theorem 3], we can see that Y is paracompact. Hence, by Theorem 3, Y has a locally countable open base, completing the proof.

Theorem 5. *Let f be a closed continuous mapping from a topological space X with a point-countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is separable and countably compact for each point y of Y. Then Y has point-countable open base if and only if any open covering of Y has a point-countable open refinement.*

Proof. As the "only if" part if obvious, we need only prove the "if" part. Let $\mathfrak{U}=\{U_\alpha\}$ be the point-countable open base of X and let y be any point of Y. Since $f^{-1}(y)$ is separable, $\sigma_y=\{U_\alpha \mid f^{-1}(y)\frown U_\alpha \neq \phi,\ U_\alpha\in\mathfrak{U}\}$ is countable [6, Lemma 2] and $f^{-1}(y)\subset \bigcup_{U\alpha\in\sigma y} U_\alpha =H^y$. Then $f^{-1}(y)\subset(H^y)_0\subset H^y$ and $(H^y)_0$ is an open inverse set because f is a closed continuous mapping. Then $\mathfrak{H}=\{f((H^y)_0)\mid y\in Y\}$ is an open covering of Y. By assumption, $\mathfrak{H}$ has a point-countable open refinement $\mathfrak{R}=\{R_\beta \mid \beta\in B\}$. For each $R_\beta\in\mathfrak{R}$, there exists a set $f((H^y)_0)$ such that $R_\beta\subset f((H^y)_0)$. Hence $f^{-1}(R_\beta)\subset(H^y)_0\subset H^y$. Let $V^\beta_\alpha=f^{-1}(R_\beta)\frown U_\alpha$ where $U_\alpha\in\sigma_y$, then $\{V^\beta_\alpha\}$ is countable. Hence the collection of unions of any finitely many sets of $\{V^\beta_\alpha\}$ is countable. Let $w^\beta_k(k=1,2,\cdots)$ denote this collection and let $\mathfrak{W}=\{f((w^\beta_k)_0)\mid k=1, 2,\cdots;\beta\in B\}$. Then $\mathfrak{W}$ is a point-countable open base of Y. In fact, let G be an open set of Y and let y be any point of G, then there exist R_β of $\mathfrak{R}$ and $(H^{y'})_0$ such that $f^{-1}(y)\subset f^{-1}(R_\beta)$ and $f^{-1}(R_\beta)\subset(H^{y'})_0$. For each point x of $f^{-1}(y)$, there exists $U_\alpha(x)\in\mathfrak{U}$ such that $U_\alpha(x)\subset f^{-1}(G)\frown f^{-1}(R_\beta)$ because $\mathfrak{U}$ is the open base. Since σ_y is countable, $\{U_\alpha(x)\mid x\in f^{-1}(y)\}$ is countable. Since $f^{-1}(y)$ is countably compact, there exists a finite subfamily of $\{U_\alpha(x)\mid x\in f^{-1}(y)\}$ which covers $f^{-1}(y)$. Let $\{U_{\alpha_i}\mid i=1,2,\cdots,k\}$ denote this subfamily, then $f^{-1}(y)\subset\bigcup_{i=1}^{k}U_{\alpha_i}\subset f^{-1}(G)\frown f^{-1}(R_\beta)\subset(H^{y'})_0$. Therefore $\left(\bigcup_{i=1}^{k}U_{\alpha_i}\right)_0$ is a set of $\{(w^\beta_k)_0\mid k=1,2,\cdots\}$. This shows that $\mathfrak{W}$ is the open base of Y. We shall next prove that $\mathfrak{W}$ is point-countable. Let y be any point of Y. Since $\{R_\beta\mid y\in R_\beta,\ R_\beta\in\mathfrak{R}\}$ is countable and $f^{-1}(R_\beta)\supset\bigcup_{k=1}^{\infty}(w^\beta_k)_0$, $\mathfrak{W}$ is point-countable. This completes the proof.

Remark 1. B. Ponomarev [6] has shown the following theorem: *if f is a closed continuous mapping from a T_1-space X with a point-countable open base onto a T_1-space Y such that the inverse image $f^{-1}(y)$ is compact for each point y of Y, then Y has a point-countable open base.* Form his proof, we see that he has used the "compactness of $f^{-1}(y)$" in the meaning of the separability and countable compactness of $f^{-1}(y)$. But it seems to us that there is a gap in his proof and we do not know whether this theorem is true or not.

Theorem 6. *If f is a closed continuous mapping from a paracompact Hausdorff (or paracompact regular topological) space X with a point-countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is separable and countably compact for each point y of Y, then Y has a point-countable open base.*

Proof. (i) The case when X is a paracompact Hausdorff space. In this case, by E. Michael's theorem [2], we can see that Y is a paracompact Hausdorff space. Then, by Theorem 5, Y has a point-countable open base.

(ii) The case when X is a paracompact regular topological space. We shall prove that $f^{-1}(y)$ is compact for each point y of Y. Let $\mathfrak{U}=\{U_\alpha\}$ be a point-countable open base of X. Since $f^{-1}(y)$ is separable, $\mathfrak{U}_y=\{U_\alpha \mid f^{-1}(y)\frown U_\alpha\neq\phi,\ U_\alpha\in\mathfrak{U}\}$ is countable [6, Lemma 2]. Let $\mathfrak{G}=\{G_\beta\}$ be any open covering of $f^{-1}(y)$, then, for each point x of $f^{-1}(y)$, there exist $G_\beta\in\mathfrak{G}$ and $U^\beta_{\alpha_x}\in\mathfrak{U}$ such that $x\in G_\beta$ and $x\in U^\beta_{\alpha_x}\subset G_\beta$. Since $\mathfrak{U}_y$ is countable and $f^{-1}(y)$ is countably compact, $f^{-1}(y)$ is covered by a finite subfamily of $\{U_{\alpha_x}\mid x\in f^{-1}(y)\}$. Let $\{U^{\beta_i}_{\alpha_i}\mid i=1, 2,\cdots, k\}$ denote this subfamily, then $f^{-1}(y)\subset\bigcup_{i=1}^{k}U^{\beta_i}_{\alpha_i}$. Therefore $f^{-1}(y)$ is covered by $\{G_{\beta_i}\mid i=1, 2,\cdots, k\}$. Thus we get the compactness of $f^{-1}(y)$. Moreover, by use of the compactness of $f^{-1}(y)$ for each point y of Y, it is easily seen that Y is regular. Then Y is paracompact [4]. By Theorem 5, we get the theorem, completing the proof.

2. Open basis and open mappings. In this section, we deal with the open basis of the image spaces of open continuous mappings.

Theorem 7. *If f is an open continuous mapping from a topological space X with a star-countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is connected for each point y of Y, then Y has a star-countable open base.*

Proof. Let $\mathfrak{U}=\{U_\alpha\}$ be a star-countable open base of X, then X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma}A_\gamma$, $A_\gamma\frown A_{\gamma'}=\phi, \gamma\neq\gamma', \gamma, \gamma'\in\Gamma$, and $A_\gamma=\smile\{U_\alpha\in\mathfrak{U}_\gamma\}$ where $\mathfrak{U}_\gamma$ is a countable subfamily of $\mathfrak{U}$. By the same argument of the proof of Theorem 1, we get $f^{-1}f(A_\gamma)=A_\gamma$ and $f(A_\gamma)\frown f(A_{\gamma'})=\phi$ for $\gamma\neq\gamma', \gamma, \gamma'\in\Gamma$. Then $\{f(U_\alpha)\mid U_\alpha\in\mathfrak{U}_\gamma\}$ is a countable open base of $f(A_\gamma)$ which is open. Therefore Y has a star-countable open base. This completes the proof.

Theorem 8. *If f is an open continuous mapping from a strongly paracompact topological space X onto a regular topological space*

Y such that the inverse image $f^{-1}(y)$ is connected for each point y of Y, then Y is strongly paracompact.

Proof. Let $\mathfrak{G}=\{G_\alpha\}$ be any open covering of Y, then $\mathfrak{H}=\{f^{-1}(G_\alpha)\mid G_\alpha\in\mathfrak{G}\}$ is an open covering of X. Since X is strongly paracompact, there exists a star-finite open covering $\mathfrak{R}=\{R_\beta\}$ as a refinement of $\mathfrak{H}$. Then, as the proof of Theorem 1, X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma}A_\gamma$, $A_\gamma\frown A_{\gamma'}=\phi$, $\gamma\neq\gamma'$, $\gamma,\gamma'\in\Gamma$, and $A_\gamma=\smile\{R_\beta\mid R_\beta\in\mathfrak{R}_\gamma\}$ where $\mathfrak{R}_\gamma$ is a countable subfamily of $\mathfrak{R}$. We get also $f^{-1}f(A_\gamma)=A_\gamma$ and $f(A_\gamma)\frown f(A_{\gamma'})=\phi$ for $\gamma\neq\gamma'$. Then, we can see that $\mathfrak{k}_\gamma=\{f(R_\beta)\mid R_\beta\in\mathfrak{R}_\gamma\}$ is a countable open covering of $f(A_\gamma)$ which is open. Let $\mathfrak{k}=\bigcup_{\gamma\in\Gamma}\mathfrak{k}_\gamma$, then, it is easy to see that $\mathfrak{k}$ is a star-countable open refinement of $\mathfrak{G}$. Since Y is a regular topological space, by Yu. Smirnov's theorem [8], Y is strongly paracompact. This completes the proof.

Theorem 9. *If f is an open continuous mapping from a locally separable metric space X onto a regular T_1-space Y such that the inverse image $f^{-1}(y)$ is connected for each point y of Y, then Y is a locally separable metric space.*

Proof. Since X is a locally separable metric space, X is strongly paracompact [3, 8]. Therefore, by Theorem 8, Y is strongly paracompact. It is shown in our previous note [1] that Y is locally separable and locally metrizable. Hence, by Nagata-Smirnov's theorem [5, 7], Y is locally separable and metrizable. This completes the proof.

Remark 2. We can prove Theorem 9 by use of A. H. Stone's theorem [9]. In fact, we can see that there exists a star-fiinite open covering $\{G_\alpha\}$ of X where each G_α is perfectly separable. Since $f^{-1}(y)$ is connected, it is easily shown that $f^{-1}(y)$ is separable. Then, by A. H. Stone's theorem, Y is a locally separable metric space.

Theorem 10. *If f is an open continuous mapping from a topological space X with a locally countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is separable for each pint y of Y, then X has a locally countable open base.*

Proof. Since X has a locally countable open base, X is locally separable. Then, since f is an open continuous mapping, Y is locally separable. Therefore, for each point y of Y, there exists an open set N such that $y\in N$ and N is separable. Since $f^{-1}(y)$ is separable for each point y of Y, $f^{-1}(N)$ is separable [9, Lemma 2]. Let $\mathfrak{U}=\{U_\alpha\}$ be a locally countable open base of X, then $\{U_\alpha\mid f^{-1}(N)\frown U_\alpha\neq\phi,\ U_\alpha\in\mathfrak{U}\}$ is countable [6, Lemma 2]. On the other hand, $\mathfrak{B}=\{f(U_\alpha)\mid U_\alpha\in\mathfrak{U}\}$ is the open base and $N\frown f(U_\alpha)\neq\phi$ if and only if $f^{-1}(N)\frown U_\alpha\neq\phi$. Therefore $\mathfrak{B}$ is a locally countable open base of Y. This completes the proof.

Remark 3. As it is easily shown that a topological space X has a star-countable open base if and only if X has a locally countable open base, we can see that the assumptions imposed on $f^{-1}(y)$ in Theorems 7 and 10 can be replaced by the separability and the connectivity respectively.

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