# 123. On the Spectra of Some Non-linear Operators. II

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In this note, we continue the study on the Hammerstein operators whose spectra contain no intervals. We denote the spectrum of a Hammerstein operator $H$ by $S(H)$.[1]

§1. Let $f_i(x)$ $(i=1,2,\cdots)$ be countable number of real-valued continuous functions with $f_i(0)=0$ defined on the whole real line, and $k_i$ $(i=1,2,\cdots)$ be countable number of positive numbers. We define an operator $H$ on $l^2$ of vectors $\phi=(x_1, x_2,\cdots)$ with $\sum_{i=1}^{\infty} x_i^2<+\infty$ by

$$H\phi=(k_1f_1(x_1), k_2f_2(x_2),\cdots). \tag{1}$$

We assume that the range of $H$ is also in $l^2$. This is of Hammerstein type, i.e. $H=K\mathfrak{f}$, where

$$\mathfrak{f}\phi=(f_1(x_1), f_2(x_2),\cdots)$$

and $K$ is a matrix of diagonal form.

*Theorem* 1. *Let us assume that the functions* $g_i(x)=\dfrac{f_i(x)}{x}$ *be continuous. Then, for the operator* $H\phi$ *defined by* (1), *if* $S(H)$ *contains no intervals,* $H$ *must be linear.*

*Proof.* When $k_1f_1(x_1)=\lambda x_1$ for some $x_1\neq 0$ and $\lambda\neq 0$, then we consider the vector $\phi_1=(x_1, 0, 0,\cdots)$ for which we have

$$H\phi_1=(k_1f_1(x_1), k_2f_2(0), k_3f_3(0),\cdots)$$
$$=(\lambda x_1, 0, 0,\cdots)=\lambda\phi_1,$$

namely, $\lambda\in S(H)$. Therefore, if the continuous function $g_1(x)$ takes two different values $\lambda_1, \lambda_2$ at points different from zero:

$$k_1g_1(x_1)=\lambda_1,\ k_1g_1(x_2)=\lambda_2;\quad x_1\neq 0,\ x_2\neq 0,$$

then, since $k_1g_1(x)$ takes every value between $\lambda_1$ and $\lambda_2$, $S(H)$ contains at least one interval. Namely, if $S(H)$ contains no intervals, $k_1g_1(x)$ must be constant, and hence it follows that

$$k_1f_1(x)=\lambda_1 x \qquad (-\infty<x<+\infty)$$

for a uniquely defined number $\lambda_1$. Similarly, we have

$$k_if_i(x)=\lambda_i(x) \qquad (-\infty<x<+\infty;\ i=2,3,\cdots).$$

Therefore, for $\phi=(x_1, x_2,\cdots)$ and $\psi=(y_1, y_2,\cdots)$, we have

$$H(x\phi+y\psi)=(k_1f_1(xx_1+yy_1), k_2f_2(xx_2+yy_2),\cdots)$$
$$=(\lambda_1(xx_1+yy_1), \lambda_2(xx_2+yy_2),\cdots)$$
$$=x(k_1f_1(x_1), k_2f_2(x_2),\cdots)+y(k_1f_1(y_1), k_2f_2(y_2),\cdots)$$

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1) As was pointed out in the preceding paper [2], we need only to study the case when $H0=0$.

$$=xH\phi+yH\psi,$$

which shows that $H$ is linear.

§2. We consider *the integral operator of Hammerstein type*[2)]

$$H\phi(s)=K\mathfrak{f}\phi(s)=\int_0^1 K(s,t)f(t,\phi(t))\,dt \qquad (2)$$

defined on $L^2$. We will apply Theorem 1 to this case. The kernel $K(s,t)$ is assumed to be positive definite, symmetric and

$$\int_0^1\int_0^1 K(s,t)^2\,ds\,dt<+\infty.$$

The function $f(t,x)$ is a Carathéodory function, namely, it is continuous with respect to $x$ in $(-\infty,+\infty)$ and measurable with respect to $t$ in $[0,1]$. We assume that the operator $\mathfrak{f}\phi(t)=f(t,\phi(t))$ is defined on the whole space $L^2$ and $f(t,0)=0$ $(0\leqq t\leqq 1)$.

*Theorem* 2. *For the Hammerstein operator defined by* (2), *let us assume that* $K(s,t)\geqq 0$ a.e. $(0\leqq s,t\leqq 1)$ *and the function* $\dfrac{\mathfrak{f}(x\phi)}{x}$ *is continuous with respect to* $x$ *in* $(-\infty,+\infty)$, *and to* $\phi\in L^2$. *Then if* $S(H)$ *contains no intervals,* $H$ *must be linear on* $K(L^2)$, *the range of* $K$.

*Proof.* We can find at most countable number of proper values $k_i$ and, to them belong, proper functions $\psi_i(t)$ of the symmetric, completely continuous, linear operator $K$. Since $K(s,t)\geqq 0$ a.e. $(0\leqq s,t\leqq 1)$, we can choose $\psi_i(t)$ as non-negative: $\psi_i(t)\geqq 0$ a.e. $(0\leqq t\leqq 1)$. The orthogonality of the proper functions implies that $\psi_i(t)\cdot\psi_j(t)=0$ $(i\neq j)$ almost everywhere, and hence it follows that $f(t,\psi_i(t))\cdot f(t,\psi_j(t))=0$ almost everywhere and that $f(t,x\psi_i(t)+y\psi_j(t))=f(t,x\psi_i(t))+f(t,y\psi_j(t))$ almost everywhere for numbers $x$ and $y$. Now, let us consider the functions

$$f_i(x)=(\mathfrak{f}(x\psi_i),\psi_i) \qquad (-\infty<x<+\infty).$$

Then, for any $\phi$ in $K(L^2)$,

$$H\phi=\sum_{i=1}^{\infty}(H\phi,\psi_i)\psi_i$$
$$=\sum_{i=1}^{\infty}k_i(\mathfrak{f}\phi,\psi_i)\psi_i,$$

and $\phi=\sum_{i=1}^{\infty}x_i\psi_i$ where $x_i=(\phi,\psi_i)$. Since we have $\mathfrak{f}\phi=\sum_{i=1}^{\infty}\mathfrak{f}(x_i\psi_i)$,

$$H\phi=\sum_{i=1}^{\infty}k_i\Big(\sum_{j=1}^{\infty}\mathfrak{f}(x_j\psi_j),\psi_i\Big)\psi_i$$
$$=\sum_{i=1}^{\infty}k_i(\mathfrak{f}(x_i\psi_i),\psi_i)\psi_i=\sum_{i=1}^{\infty}k_if_i(x_i)\psi_i.$$

Therefore, by Theorem 1, $H$ must be linear on $K(L^2)$.

2) For the detail, we refer Chapter I of [1].

## References

[1] M. A. Krasnoseliski: Topological Method in the Theory of Non-linear Integral Equations, Moscow (1956).
[2] S. Yamamuro: On the spectra of some non-linear operators. I, Proc. Japan Acad., **37**, 447-451 (1961).