145. Approximation of Solutions of Homogeneous Differential Equation. I

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§ 1. Hereafter we will use these notations:

$$P(D)\,U=0, \tag{1}$$

where $D=(D_1,\cdots,D_n)=\left(\frac{1}{i}\partial_{x_1},\cdots,\frac{1}{i}\partial_{x_n}\right)$, a homogeneous differential equation with constant coefficient,

$C^n=C\times C\times\cdots\times C$, a direct product of N complex planes,

$R^n=R\times R\times\cdots\times R$, a direct product of N real ax'es,

(D), the set of functions $U\in C^\infty$ with compact carrier,

(D'), the set of functionals on (D) in the sense of L. Schwartz.

We begin with a definition:

Definition 1. A solution of (1) in R^n is called an exponential solution if it can be written in the form

$$U(X)=f(X)\,e^{i\langle X\,\zeta\rangle} \tag{2}$$

where $\zeta\in C^n$ and $f(X)$ is a polynomial.

Approximation of solutions of (1) by the set of exponential solutions has been discussed by Lars Hörmander and B. Malgrange, etc. The following theorem is evident from Hörmander's result.

Theorem 1. The closed linear bull in (D') of the exponential solutions of (1) consists of all solutions of (1) in (D').

In the following, we will give an approximation of solutions of (1) by the smaller set of solutions which consist of complex linear combinations of exponential solutions.

Namely these solutions are written in the form

$$U(X)=e^{i\langle X_1\zeta_1+X_2\xi_2+\cdots+X_n\xi_n\rangle}, \tag{3}$$

where $\zeta_1\in C$ and $(\xi_2,\cdots,\xi_n)\in R^{n-1}\frown U(\xi_2^0,\cdots,\xi_n^0)$ for an arbitrary fixed neighbourhood $U(\xi_2^0,\cdots,\xi_n^0)$ of an arbitrary fixed point $(\xi_2^0,\cdots,\xi_n^0)\in R^{n-1}$.

We have already found the applications of this result in many different directions. Without proof we shall show the result which we have gotten, and we shall show the outline of the proof with an example. Afterwards we shall treat system's case in (II) and (III).

§ 2. Let's consider the function $U(X)=f(X)\,e^{i\langle X\zeta\rangle}$ where $\zeta\in C^n$ and $f(X)$ is a polynomial.

Lemma 1. If $U(X)$ is the solution of the equation $P(D)U=0$, then $P^{(\alpha)}(\zeta)=0$ for α by which $D_\alpha\,f(X)\neq 0$.

Let's consider the polynomial with complex coefficient, $P(\zeta)=a_0\zeta_1^k+a_1(\zeta_2,\cdots,\zeta_n)\zeta_1^{k-1}+\cdots+a_k(\zeta_1,\cdots,\zeta_n)$.

Because by a rotation in C^n, every $P(\zeta)$ can take this form. Accordingly we can decide the root of the equation $P(\zeta)=0$; $\lambda_1(\zeta_2,\cdots,\zeta_n)$, $\lambda_2(\zeta_2,\cdots,\zeta_n),\cdots,\lambda_k(\zeta_2,\cdots,\zeta_n)$.
For the sake of simplicity we shall write $\exp(i\lambda_i(\zeta_2,\cdots,\zeta_n)X_1+i\zeta_2X_2+\cdots+i\zeta_nX_n)=E_i(\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n)=E(\lambda i(\zeta_2,\cdots,\zeta_n),\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n)$ and $S_i(R^{n-1}\frown U(\xi_2^0,\cdots,\xi_n^0))=\{E_i(\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n);(\zeta_2,\cdots,\zeta_n)\in R^{n-1}\frown U(\xi_2^0,\cdots,\xi_n^0)\}$.

In Lemma 2 and 4 we will use the following notation:

$$P(\zeta)\ P_k(\zeta)(1\leq k\leq m)$$

are polynomials of $\zeta_1,\cdots,\zeta_n$ with complex coefficient, and $A(\zeta_2,\cdots,\zeta_n)$ is a polynomial of $\zeta_2,\cdots,\zeta_n$ with complex coefficient.

Lemma 2. If $P(\zeta)=0$ has a ρ-ple root of $(2\leq\rho\leq k)$, namely

$$\lambda_i(\zeta_2,\cdots,\zeta_n)\equiv\cdots\equiv\lambda_{i+\rho-1}(\zeta_2,\cdots,\zeta_n),$$

then for a suitable polynomial $A(\zeta_2,\cdots,\zeta_n)$, $P(\zeta)$ takes the following form: $A(\zeta_2,\cdots,\zeta_n)\,P(\zeta)=P_1(\zeta)^{n_1}\cdots P_m(\zeta)^{n_m}$, where $n_1,\cdots,n_m$ are not all 1.

Lemma 3. If $\lambda_i(\zeta_2,\cdots,\zeta_n)$ is not a ρ-ple root $(\rho\geqq 2)$ in the neighbourhood $U(\zeta_2^0,\cdots,\zeta_n^0)$, then $\lambda_i(\xi_2,\cdots,\xi_n)$ is an analytic function in the neighbourhood $\tilde{U}(\zeta_2^0,\cdots,\zeta_n^0)\subset U(\zeta_2^0,\cdots,\zeta_n^0)$.

Lemma 4. Let's consider the case in which $P(\zeta)$ does not take the following form:

$A(\zeta_2,\cdots,\zeta_n)\,P(\zeta)=P_1(\zeta)^{n_1}\cdots P_m(\zeta)^{n_m}$, where $n_1,\cdots,n_m$ are not all 1. Then $\{(\zeta_2,\cdots,\zeta_n);\lambda_j(\zeta_2,\cdots,\zeta_n)=\cdots=\lambda_{j+\rho-1}(\zeta_2,\cdots,\zeta_n)\}\subseteq\{(\zeta_2,\cdots,\zeta_n);\tilde{A}(\zeta_2,\cdots,\zeta_n)=0\}$, where $\tilde{A}(\zeta_2,\cdots,\zeta_n)$ is a polynomial with complex coefficient.

Using the method like the Euclidean algorithm, we can prove Lemma 2 and 4.

Lemma 5. If $\lambda_i(\zeta_2,\cdots,\zeta_n)\equiv\lambda_j(\zeta_2,\cdots,\zeta_n)$ in an open domain, then $\lambda_i(\zeta_2,\cdots,\zeta_n)\equiv\lambda_j(\zeta_2,\cdots,\zeta_n)$ in the whole complex space.

Lemma 6. Let's consider the functions $E(\lambda_i,\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n)$ $(i\leq j\leq i+\rho-1)$, where $\lambda_i(t\zeta_2^0,\cdots,t\zeta_n^0)=\lambda_j(t\zeta_2^0,\cdots,t\zeta_n^0)\ (i\leq j\leq i+\rho-1)$ at only one point $t=0$.

Then we can approximate $X_1^lE(\lambda_i(0,\cdots,0),0,\cdots,0,X_1,\cdots,X_n)$ $(1\leq l\leq\rho-1)$ by the complex linear combinations of $E(\lambda_j,\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n)\,(i\leq j\leq i+\rho-1)$ in (D').

From the equidifferentiability of $E(\zeta_1,\zeta_2,\cdots,\zeta_n,X_1^0,\cdots,X_n^0)$ in a compact set and Heine Borel's Theorem, we can prove this lemma easily.

Lemma 7. $E(\lambda_i,\zeta_2,\cdots,\zeta_n,X_1,\cdots,X_n)$ is in the linear bull of $S_i(U(\xi_2^0,\cdots,\xi_n^0)\frown R^{n-1})$ in (D').

From the uniform convergence of power series and the finite difference form we can prove this lemma.

Theorem 2. If $P(\zeta)=0$ has not a multiple root, namely $P(\zeta)$ does not take the following form $A(\zeta_2,\cdots,\zeta_n)\,P(\zeta)=P_1(\zeta)^{n_1}\cdots P_m(\zeta)^{n_m}$, where $A(\zeta_2,\cdots,\zeta_n)$, $P(\zeta)$, $P_1(\zeta),\cdots,P_m(\zeta)$ are polynomials, and $n_1,\cdots,n_m$ are not all 1;
then the closed linear bull in (D') of $\bigcup\limits_i S_i(U_{(i)}(\xi_2^{0(i)},\cdots,\xi_n^{0(i)})\frown R^{n-1})$ consists of all solutions of $P(D)U=0$ in (D').

Theorem 3. If $P(\zeta)=0$ has a multiple root, namely

$$\lambda_j(\zeta_2,\cdots,\zeta_n)\equiv\cdots\equiv\lambda_{j+\rho-1}(\zeta_2,\cdots,\zeta_n),$$

then the closed linear bull in (D') of

$$\Big[\bigcup_i\{S_i(U_{(i)}(\xi_2^{0(i)},\cdots,\xi_n^{0(i)})\frown R^{n-1})\}\Big]\smile\Big[\bigcup_{1\le k\le\rho-1}\{X_1^k e^{i\lambda_j X_1+\cdots+i\xi_n X_n};\ (\xi_2,\cdots,\xi_n)\in U_{(j)}(\xi_2^{0(j)},\cdots,\xi_n^{0(j)})\frown R^{n-1}\}\Big]$$

consists of all solutions of $P(D)U=0$ in (D').

§3. An Example:

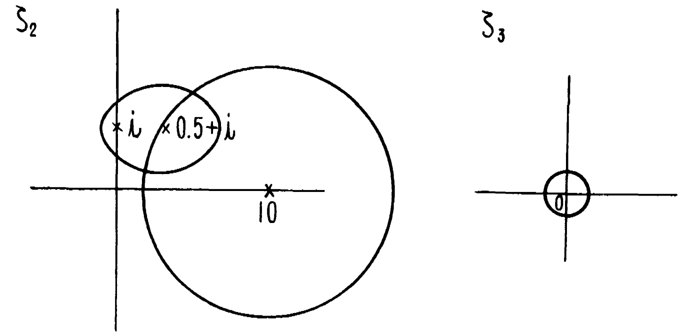
$$\partial_x^2U+\partial_y^2U+\partial_z^2U=0,\ \ -\zeta_1^2-\zeta_2^2-\zeta_3^2=0,\ \ \zeta_1=\pm i\sqrt{\zeta_2^2+\zeta_3^2}.$$

$$e^{-\sqrt{\zeta_2^2+\zeta_3^2}\,x+i\zeta_2y+i\zeta_3z},\ e^{\sqrt{\zeta_2^2+\zeta_3^2}\,x+i\zeta_2y+i\zeta_3z}.$$

$$\zeta_2=\pm i\zeta_3,\ (i\zeta_3,\zeta_3).$$

$$\zeta_2=10,\ \zeta_3=0\to\zeta_1=\pm10i.$$

$$\zeta_2=i,\ \ \zeta_3=0\to\zeta_1=\mp i.$$



$$e^{-ix-y}=\sum_{p_2=0}^{k_2}\sum_{p_3=0}^{k_3}\Big[\partial_{\zeta_2}^{p_2}\partial_{\zeta_3}^{p_3}\sum_{p'_2=0}^{k'_2}\sum_{p'_3=0}^{k'_3}\Big\{\Big(\frac{\tau_{h_2}-1}{h_2}\Big)^{p'_2}\Big(\frac{\tau_{h_3}-1}{h_3}\Big)^{p'_3}e^{-\sqrt{\xi_2^2+\xi_3^2}\,x+i\xi_2y+i\xi_3z}\Big\}_{\xi_2=10,\xi_3=0}$$

$$\frac{(\zeta_2-10)^{p'_2}}{p'_2!}\frac{(\zeta_3-0)^{p'_3}}{p'_3!}\Big]_{\substack{\zeta_2=0.5+i\\ \zeta_3=0}}\frac{(-0.5)^{p_2}}{p_2!}\frac{0^{p_3}}{p_3!}\quad(0^0\equiv1)$$

$$\lim_{h\to0}\frac{e^{-\sqrt{(i\zeta_3+h)^2+\zeta_3^2}\,x+i(i\zeta_3+h)y+i\zeta_3z}-e^{\sqrt{(i\zeta_3+h)^2+\zeta_3^2}\,x+i(i\zeta_3+h)y+i\zeta_3z}}{i\sqrt{(i\zeta_3+h)^2+\zeta_3^2}+i\sqrt{(i\zeta_3+h)^2+\zeta_3^2}}=-\frac{X}{i}e^{-\zeta_3y+i\zeta_3z}.$$

$$((2i\zeta_3+h)h\neq0\quad h\neq-2i\zeta_3)$$

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