144. Cohomology Mod 2 of the Compact Exceptional Group E_8

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1. By $\boldsymbol{E}_8$ we mean a compact exceptional Lie group whose local structure is usually expressed by the same letter. It is unique up to isomorphisms. In this note we determine the cohomology ring mod 2 of $\boldsymbol{E}_8$. The result is as follows:

Theorem 1. $H^*(\boldsymbol{E}_8; Z_2)=Z_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4)$
$\otimes\Lambda_2(x_{17}, x_{23}, x_{27}, x_{29})$
where the suffix of each generator denotes its degree.

Thus $H^*(\boldsymbol{E}_8; Z_2)$ is a truncated polynomial ring over Z_2 with 8 generators of degrees 3, 5, 9, 15, 17, 23, 27 and 29 respectively, and the heights of generators are 16, 8, 4, 4, 2, 2, 2, 2.

The relations among generators by Steenrod squares are as follows:

Theorem 2. *In the above theorem, the generators of* $H^*(\boldsymbol{E}_8; Z_2)$ *can be chosen to satisfy the relations*:

$$x_5=Sq^2x_3,\ x_9=Sq^4x_5,\ x_{17}=Sq^8x_9,$$
$$x_{23}=Sq^8x_{15},\ x_{27}=Sq^4x_{23},\ x_{29}=Sq^2x_{27}.$$

It is still open whether $Sq^2x_{15}=0$ or x_{17}.

2. Further we obtain the

Proposition 1. $\boldsymbol{E}_7$ *is totally non-homologous zero* mod 2 *in* $\boldsymbol{E}_8$.

In this proposition $\boldsymbol{E}_7$ means the compact simply-connected exceptional group with the local structure expressed by the same letter. This proposition, combined with the prop. 22.4 of Borel [4] and the props. 2.8, 3.12 of [3], proves the

Theorem 3. *In the inclusions* $\boldsymbol{G}_2\subset\boldsymbol{F}_4\subset\boldsymbol{E}_6\subset\boldsymbol{E}_7\subset\boldsymbol{E}_8$, *every subgroup is totally non-homologous zero* mod 2 *in any bigger group containing it, where each exceptional group denotes the compact simply-connected one.*

This theorem holds only for homologies "mod 2".

In [3, 4] the cohomology rings mod 2 of the first four simply-connected exceptional groups are calculated. We shall restate them here.

(1) $H^*(\boldsymbol{G}_2; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5)$,

(2) $H^*(\boldsymbol{F}_4; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5, x_{15}, x_{23})$,

(3) $H^*(\boldsymbol{E}_6; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5, x_9, x_{15}, x_{17}, x_{23})$,

(4) $H^*(\boldsymbol{E}_7; Z_2)=Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4)\otimes\Lambda_2(x_{15}, x_{17}, x_{23}, x_{27})$,

where the suffix of each generator denotes the degree. Furthermore relations among generators by squaring operations are determined as follows:

$$(5)\qquad \begin{aligned} &x_5=Sq^2x_3 &&\text{for } \boldsymbol{G}_2, \boldsymbol{F}_4, \boldsymbol{E}_6, \boldsymbol{E}_7,\\ &x_9=Sq^4x_5,\ x_{17}=Sq^8x_9 &&\text{for } \boldsymbol{E}_6, \boldsymbol{E}_7. \end{aligned}$$

Theorems 2 and 3 conclude immediately the following new relations:

Theorem 4. *Generators of the cohomology rings of* (2)–(4) *can be chosen to satisfy the following relations*:

$$x_{23}=Sq^8x_{15} \text{ for } \boldsymbol{F}_4, \boldsymbol{E}_6, \boldsymbol{E}_7,$$
$$x_{27}=Sq^4x_{23} \text{ for } \boldsymbol{E}_7$$

besides the relations (5).

3. By discussing the spectral sequence mod 2 of the covering $\boldsymbol{E}_7 \to \boldsymbol{Ad}\,\boldsymbol{E}_7$, (4), (5) and Theorem 4 allow us to obtain the

Theorem 5. $H^*(\boldsymbol{Ad}\,\boldsymbol{E}_7; Z_2)=Z_2[x_1, x_3, x_5, x_9]/(x_1^{16}, x_3^4, x_5^4, x_9^4) \otimes \Lambda_2(x_{17}, x_{23}, x_{27})$.

Thus, (1)–(4), Theorems 1 and 5 determine completely the cohomology ring mod 2 of all compact exceptional groups. Hence we know now the cohomology rings mod p of all compact exceptional groups for every prime p, via Borel [4, 5, 6] and [2].

4. For any topological space X we denote the Bockstein cohomology spectral sequences mod p of X by $E_r(X; Z_p)$, $0\leq r\leq\infty$ and $E_0(X; Z_p)=H^*(X; Z_p)$. By [2, 3, 4, 5, 6] and Theorem 1 the cohomology rings mod p of all compact simply-connected simple Lie groups are determined for every prime p, including the Bockstein operations mod p. Hence the Bockstein spectral sequences of each simply-connected simple groups can be discussed now.

The E_r terms of Bockstein spectral sequences of any compact simply connected Lie group G are the tensor products of the corresponding terms of each simple factors (which is simply-connected) by the Künneth's formula. In this way we prove that $E_1(G; Z_p) = E_\infty(G; Z_p)$ for every prime p. This is interpreted as in the following

Theorem 6. *For any compact simply-connected Lie group G, the p-primary component of the torsion of the integral cohomology group of G is a direct sum of a finite number of copies of Z_p for every prime p.*

These phaenomena were observed and remarked by A. Borel about several simple groups.

5. To prove Theorem 1 we first discuss the cohomology mod 2 of $\Omega(\boldsymbol{E}_8/\boldsymbol{E}_7\times\boldsymbol{A}_1)$ in low degrees according to the scheme of Bott-Samelson [7], where $\boldsymbol{E}_8/\boldsymbol{E}_7\times\boldsymbol{A}_1$ is the compact simply-connected symmetric space of type (EIX) by the notation of E. Cartan [8]. Then we obtain the

Proposition 2. $H^*(\Omega(\boldsymbol{E}_8/\boldsymbol{E}_7\times\boldsymbol{A}_1); Z_2)=Z_2[u_1]/(u_1^4) \otimes\Lambda_2(u_{11}, u_{19}, u_{28}, u_{29})$ *in degrees* $\leqq 29$.

Discussing similarly as in the prop. 3.8 of [3], we conclude from the above proposition the following

Proposition 3. $H^*(\Omega(E_8/E_7); Z_2) = \Lambda_2(u_{11}, u_{19}, u_{28}, u_{29})$ *in degrees* $\leqq 29$.

Some discussions of K-cycles of $\Omega(E_8/E_7 \times A_1)$ conclude the following relations among generators of the above Props. 2 and 3:

(6) $$u_{19} = Sq^8 u_{11} \text{ and } u_{29} = Sq^1 u_{28}.$$

Discussing the spectral sequence mod 2 of the fibration $(\Omega E_8, \Omega(E_8/E_7), \Omega E_7)$ by making use of the Prop. 3, we get the following:

In any choice of the generator u_{14} *of degree* 14 *of* $H^*(\Omega E_8; Z_2)$, $(u_{14})^2 \neq 0$. *For the generator* u_2 *of degree* 2 *of* $H^*(\Omega E_8; Z_2)$, $(u_2)^n \neq 0$ *for* $n \leqq 15$.

By Adem relations on squaring operations [1.9]

$$Sq^{14} = Sq^2 Sq^{12} + Sq^{13} Sq^1$$
$$= Sq^2 Sq^4 Sq^8 + (Sq^2 Sq^{11} + Sq^{13}) Sq^1 + Sq^2 Sq^{10} Sq^2.$$

Clearly $Sq^1 u_{14} = 0$ and $Sq^2 u_{14} = 0$ or $(u_2)^8$ in $H^*(\Omega E_8; Z_2)$. In any case $Sq^{10} Sq^2 u_{14} = 0$. Hence

$$(u_{14})^2 = Sq^{14} u_{14} = Sq^2 Sq^4 Sq^8 u_{14}.$$

In particular $Sq^8 u_{14} \neq 0$, $Sq^4 Sq^8 u_{14} \neq 0$ in $H^*(\Omega E_8; Z_2)$. Therefore we can choose the generators u_{22} and u_{26} of $H^*(\Omega E_8; Z_2)$ to satisfy the equalities

$$u_{22} = Sq^8 u_{14} \text{ and } u_{26} = Sq^4 u_{22}.$$

And we have the

Proposition 4. $H^*(\Omega E_8; Z_2) = Z_2[u_2, u_{14}, u_{22}, u_{26}]$ *in degrees* $\leqq 31$ *with relations* $u_{22} = Sq^8 u_{14}$ *and* $u_{26} = Sq^4 u_{22}$.

In the spectral sequence mod 2 of the fibration of loop spaces of E_8 the generator u_{14} of the fibre cohomology can be chosen to be transgressive. Then we can determine $H^*(E_8; Z_2)$ in degrees $\leqq 32$ from the above proposition to coincide with the assertion of Theorems 1 and 2.

Finally a dimensionality argument as in [3, 6] completes the proof of our Theorem 1.

6. In the spectral sequence mod 2 of the fibration of loop spaces of E_8/E_7 the generators described in the Prop. 3 are transgressive. Hence we have the

Proposition 5. $H^*(E_8/E_7; Z_2) = Z_2[y_{12}, y_{20}, y_{29}, y_{30}]$ *in degrees* $\leqq 30$ *with relations*: $y_{20} = Sq^8 y_{12}$ *and* $y_{30} = Sq^1 y_{29}$.

Now a discussion of the spectral sequence mod 2 of the fibration $(E_8, E_8/E_7, E_7)$, by making use of the Prop. 5 and Theorem 1, concludes the Prop. 1.

References

[1] J. Adem: The iteration of the Steenrod squares in algebraic topology, Proc. Nat. Acad. Sci., U.S.A., **38**, 720–726 (1952).

[2] S. Araki: Differential Hopf algebras and the cohomology mod 3 of the compact exceptional groups E_7 and E_8, Ann. of Math., **73**, 404-436 (1961).

[3] ——: Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, J. of Math. Osaka City Univ., **12** (1961), to appear.

[4] A. Borel: Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math., **76**, 272-342 (1954).

[5] ——: Sur la torsion des groupes de Lie, Jour. Math. pur. appl., **35**(9), 127-139 (1956).

[6] ——: Sousgroupes commutatifs et torsion des groupes de Lie compactes connexes, Tohoku Math. J., **13**(2), 216-240 (1961).

[7] R. Bott and H. Samelson: Applications of the theory of Morse to symmetric spaces, Amer. J. Math., **80**, 964-1029 (1958).

[8] E. Cartan: Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, Ann. Sci. de l'Ec. Norm. Sup., **44**, 345-467 (1927).

[9] H. Cartan: Sur l'itération des opérations de Steenrod, Comment. Math. Helvet., **29**, 40-58 (1955).