143. Functional-Representations of Normal Operators in Hilbert Spaces and their Applications

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In this paper we have mainly two aims: one is to express a normal operator in a Hilbert space by continuous linear functionals associated with all elements of a complete orthonormal set in that space and the other is to construct a normal operator with the arbitrarily prescribed point spectrum. We can yet treat these two problems at the same time.

Definition. Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ both be incomplete orthonormal infinite sets which have no element in common and together form a complete orthonormal set in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed bounded sequence in the complex plane; let (u_{ij}) be an infinite unitary matrix with $|u_{jj}|\neq 1, j=1,2,3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let N be the operator defined by

$$Nx=\sum_{j=1}^{\infty}\lambda_\nu(x, \varphi_\nu)\varphi_\nu+c\sum_{\mu=1}^{\infty}(x, \psi_\mu)\Psi_\mu$$

for every $x\in\mathfrak{H}$ and an arbitrarily given constant c; let L_f be the continuous linear functional associated with an arbitrary element f in $\mathfrak{H}$; and let the operator N and the element Nx, defined above, be denoted symbolically by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu} \tag{1}$$

and

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x) \tag{2}$$

respectively. Then the sum of the two series in the right-hand side of (1) is called "the functional-representation of the operator N".

Theorem 1. The functional-representation of the operator N defined by (1) converges uniformly and N is a bounded normal operator with the point spectrum $\{\lambda_\nu\}$ on $\mathfrak{H}$. In addition, putting $M=\max(S, |c|^2)$ where $S=\sup_\nu|\lambda_\nu|^2$, $||N||=\sqrt{M}$.

Proof. Since, by hypotheses, a complete orthonormal set is formed by the two sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$, we have for every $x\in\mathfrak{H}$

$$x=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu,$$

where $a_\nu = L_{\varphi_\nu}(x)$ and $b\mu = L_{\psi_\mu}(x)$. On the other hand, since, by hypotheses, (u_{ij}) is an infinite unitary matrix,

$$\sum_{j=1}^{\infty} u_{\mu j}\bar{u}_{\kappa j} = \begin{cases} 1 & (\mu=\kappa) \\ 0 & (\mu \neq \kappa) \end{cases}$$

and

$$\sum_{i=1}^{\infty} u_{i\mu}\bar{u}_{i\kappa} = \begin{cases} 1 & (\mu=\kappa) \\ 0 & (\mu \neq \kappa) \end{cases}.$$

In addition, since $\|x\|^2 = \sum_{\nu=1}^{\infty}|a_\nu|^2 + \sum_{\mu=1}^{\infty}|b_\mu|^2 < \infty$, there exists a positive integer P such that, for an arbitrarily given $\varepsilon > 0$,

$$\sum_{\nu=P}^{\infty}|a_\nu|^2 + \sum_{\mu=P}^{\infty}|b_\mu|^2 < \varepsilon\|x\|^2/M, \ (x \neq 0),$$

where M is the constant defined in the statement of the present theorem.

In consequence, it is easily verified by direct computations that

$$\begin{aligned} &\left\|\sum_{\nu=P}^{\infty}\lambda_\nu\varphi_\nu \otimes L_{\varphi_\nu}(x) + c\sum_{\mu=P}^{\infty}\Psi_\mu \otimes L_{\psi_\mu}(x)\right\|^2 \\ &= \sum_{\nu=P}^{\infty}|\lambda_\nu a_\nu|^2 + |c|^2\sum_{\mu=P}^{\infty}|b_\mu|^2 \\ &< \varepsilon\|x\|^2. \end{aligned}$$

This result shows that

$$\left\|\sum_{\nu=P}^{\infty}\lambda_\nu\varphi_\nu \otimes L_{\varphi_\nu} + c\sum_{\mu=P}^{\infty}\Psi_\mu \otimes L_{\psi_\mu}\right\| < \sqrt{\varepsilon}$$

and hence that the functional-representation of N converges uniformly.

Similarly we have

$$\|Nx\|^2 \leqq M\left(\sum_{\nu=1}^{\infty}|a_\nu|^2 + \sum_{\mu=1}^{\infty}|b_\mu|^2\right) = M\|x\|^2 < \infty$$

for every $x \in \mathfrak{H}$. Hence N is a bounded operator.

Moreover, when $M=S$ we can easily verify by putting $x=\varphi_\nu$ that $\|N\|$ equals $\sqrt{M}$, whereas when $M=|c|^2$ we can show by setting $x=\psi_\kappa$ the validity of the relation $\|N\psi_\kappa\|^2 = |c|^2\|\psi_\kappa\|^2$ which implies that $\|N\|$ is equal to $\sqrt{M}$.

Next we shall prove that N is normal.

Since the identity operator I and any $y \in \mathfrak{H}$ are expressible in the forms

$$I = \sum_{\nu=1}^{\infty}\varphi_\nu \otimes L_{\varphi_\nu} + \sum_{\mu=1}^{\infty}\psi_\mu \otimes L_{\psi_\mu}$$

and

$$y = \sum_{\nu=1}^{\infty}\varphi_\nu \otimes L_{\varphi_\nu}(y) + \sum_{\mu=1}^{\infty}\psi_\mu \otimes L_{\psi_\mu}(y)$$

respectively, it is a matter of simple manipulations to show that

$$\begin{aligned} (Nx, y) &= \sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\overline{L_{\varphi_\nu}(y)} + c\left(\sum_{\mu=1}^{\infty}\Psi_\mu \otimes L_{\psi_\mu}(x), \sum_{\mu=1}^{\infty}\psi_\mu \otimes L_{\psi_\mu}(y)\right) \\ &= \sum_{\nu=1}^{\infty}\lambda_\nu L_{\varphi_\nu}(x)\overline{L_{\varphi_\nu}(y)} + c\sum_{\mu=1}^{\infty}\sum_{\kappa=1}^{\infty}u_{\mu\kappa}L_{\psi_\mu}(x)\overline{L_{\psi_\kappa}(y)}, \end{aligned} \tag{3}$$

where $\overline{L_{\psi_\kappa}(y)}$ and $\overline{L_{\varphi_\nu}(y)}$ denote the conjugate complex numbers of $L_{\varphi_\nu}(y)$ and $L_{\psi_\kappa}(y)$ respectively.

We now put

$$\Psi_\mu^* = \sum_{i=1}^{\infty} \bar{u}_{i\mu} \psi_i$$

and consider the operator $\bar{N}$ defined by

$$\bar{N} = \sum_{\nu=1}^{\infty} \bar{\lambda}_\nu \varphi_\nu \otimes L_{\varphi_\nu} + \bar{c} \sum_{\mu=1}^{\infty} \Psi_\mu^* \otimes L_{\psi_\mu}.$$

Then, by reasoning exactly like that applied to the series of the right-hand side of (1), we can prove that the above functional-representation of $\bar{N}$ converges uniformly. Moreover, in the same manner as that used to show (3), we can show the validity of the relation

$$(4) \qquad (x, \bar{N}y) = \sum_{\nu=1}^{\infty} \lambda_\nu L_{\varphi_\nu}(x) \overline{L_{\varphi_\nu}(y)} + c \sum_{\mu=1}^{\infty} \sum_{\kappa=1}^{\infty} u_{\mu\kappa} L_{\psi_\mu}(x) \overline{L_{\psi_\kappa}(y)}$$

for every pair of $x, y \in \mathfrak{H}$.

From the relations (3) and (4), it follows at once that the adjoint operator N^* of N is given by $\bar{N}$.

Furthermore,

$$\begin{aligned} NN^*x &= N\Big[\sum_{\nu=1}^{\infty} \bar{\lambda}_\nu \varphi_\nu \otimes L_{\varphi_\nu}(x) + \bar{c} \sum_{\mu=1}^{\infty} \Psi_\mu^* \otimes L_{\psi_\mu}(x)\Big] \\ &= \sum_{\nu=1}^{\infty} |\lambda_\nu|^2 \varphi_\nu \otimes L_{\varphi_\nu}(x) + |c|^2 \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} L_{\psi_\kappa}(x) L_{\psi_\mu}(\Psi_\kappa^*)\Big\} \Psi_\mu \\ &= \sum_{\nu=1}^{\infty} |\lambda_\nu|^2 \varphi_\nu \otimes L_{\varphi_\nu}(x) + |c|^2 \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} \bar{u}_{\mu\kappa} L_{\psi_\kappa}(x)\Big\} \Psi_\mu \end{aligned}$$

and

$$\begin{aligned} N^*Nx &= N^*\Big[\sum_{\nu=1}^{\infty} \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}(x) + c \sum_{\mu=1}^{\infty} \Psi_\mu \otimes L_{\psi_\mu}(x)\Big] \\ &= \sum_{\nu=1}^{\infty} |\lambda_\nu|^2 \varphi_\nu \otimes L_{\varphi_\nu}(x) + |c|^2 \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} L_{\psi_\kappa}(x) L_{\psi_\mu}(\Psi_\kappa)\Big\} \Psi_\mu^* \\ &= \sum_{\nu=1}^{\infty} |\lambda_\nu|^2 \varphi_\nu \otimes L_{\varphi_\nu}(x) + |c|^2 \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} u_{\kappa\mu} L_{\psi_\kappa}(x)\Big\} \Psi_\mu^* \end{aligned}$$

for every $x \in \mathfrak{H}$. On the other hand, it is seen with the aid of the previously described relations between the u's that

$$\begin{aligned} \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} \bar{u}_{\mu\kappa} L_{\psi_\kappa}(x)\Big\} \Psi_\mu &= \sum_{\mu=1}^{\infty} \Big\{\Big[\sum_{\kappa=1}^{\infty} \bar{u}_{\mu\kappa} L_{\psi_\kappa}(x)\Big]\Big[\sum_{i=1}^{\infty} u_{\mu i} \psi_i\Big]\Big\} \\ &= \sum_{\kappa=1}^{\infty} \psi_\kappa \otimes L_{\psi_\kappa}(x), \end{aligned}$$

$$\begin{aligned} \sum_{\mu=1}^{\infty} \Big\{\sum_{\kappa=1}^{\infty} u_{\kappa\mu} L_{\psi_\kappa}(x)\Big\} \Psi_\mu^* &= \sum_{\mu=1}^{\infty} \Big\{\Big[\sum_{\kappa=1}^{\infty} u_{\kappa\mu} L_{\psi_\kappa}(x)\Big]\Big[\sum_{i=1}^{\infty} \bar{u}_{i\mu} \psi_i\Big]\Big\} \\ &= \sum_{\kappa=1}^{\infty} \psi_\kappa \otimes L_{\psi_\kappa}(x). \end{aligned}$$

Applying the last two results to the expansions of NN^*x and N^*Nx, we have the relation $NN^*x = N^*Nx$ holding for every $x \in \mathfrak{H}$. Thus N is a normal operator.

It remains only to prove that the set $\{\lambda_\nu\}$ is the point spectrum of N. As a first step it is, however, clear that $N\varphi_\nu=\lambda_\nu\varphi_\nu$ for $\nu=1,2,3,\cdots$.

We suppose, contrary to what we wish to prove, that N has an eigenvalue α different from $\lambda_\nu, \nu=1,2,3,\cdots$, and denote by K_α the eigenprojector of N corresponding to the eigenvalue α. Since every eigenelement of N for α is orthogonal to all elements of $\{\varphi_\nu\}$, it is expressed by a linear combination of elements belonging to $\{\psi_\mu\}$. In consequence, we may and do denote an arbitrary eigenelement f_α of N corresponding to the eigenvalue α by $\sum_p a_p\psi'_p, \psi'_p\in\{\psi_\mu\}$. Then, by means of the relations $K_\alpha f_\alpha=f_\alpha$ and $((I-K_\alpha)\psi'_p, \varphi_\nu)=0, \nu=1,2,3,\cdots$, and of Parseval's formula, we have

$$0=\|\sum_p a_p(I-K_\alpha)\psi'_p\|^2$$
$$=\sum_{\mu=1}^{\infty}|\sum_p a_p((I-K_\alpha)\psi'_p, \psi_\mu)|^2,$$

so that

$$\sum_p a_p((I-K_\alpha)\psi'_p, \psi_\mu)=0, \ \mu=1,2,3,\cdots.$$

Moreover, since the eigenspace of N corresponding to the eigenvalue α is given by $K_\alpha\mathfrak{H}$ and hence since the final velations always hold for all systems of the coefficients a_p as far as $\sum_p|a_p|^2<\infty$, $(I-K_\alpha)\psi'_p$ vanishes, that is, $K_\alpha\psi'_p=\psi'_p$ for every admissible p. As a result, we find that

$$\alpha\psi'_p=N\psi'_p$$
$$=c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}(\psi'_p)$$
$$=c\sum_{\kappa=1}^{\infty}u_{j\kappa}\psi_\kappa, \ (|u_{jj}|\neq 1),$$

where j is uniquely determined by the condition $\psi_j=\psi'_p$. This result, however, is incompatible with the linear independence of ψ_κ, $\kappa=1,2,3,\cdots$, and hence the supposition concerning α must be rejected.

Thus the set $\{\lambda_\nu\}$ gives the point spectrum of N, as we were to prove.

With these results, the proof of the present theorem is complete.

Theorem 2. In Theorem 1, let $\lambda_1=\lambda_2=\cdots=\lambda_m\in\{\lambda_\nu\}$ under the condition that λ_m be different from any λ_κ for $\kappa=m+1, m+2,\cdots$; let K_ν be the eigenprojector corresponding to any eigenvalue λ_ν of N; and let $\{K(z)\}$ and Δ be the complex spectral family and the continuous spectrum of N respectively. Then

$$K_m=\sum_{j=1}^{m}\varphi_j\otimes L_{\varphi_j}, \tag{5}$$

$$\int_\Delta z\,dK(z)=c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\phi_\mu}. \tag{6}$$

Proof. By hypotheses, it follows at once that

$$K_m\varphi_j=\begin{cases}\varphi_j & (j=1,2,\cdots,m)\\ 0 & (j=m+1,m+2,\cdots),\end{cases}$$

while $K_m\psi_\mu=0$ for $\mu=1,2,3,\cdots$. Hence, by the use of the relation $x=\sum_{\nu=1}^{\infty}\varphi_\nu\otimes L_{\varphi_\nu}(x)+\sum_{\mu=1}^{\infty}\psi_\mu\otimes L_{\psi_\mu}(x)$ holding for every $x\in\mathfrak{H}$, we have $K_m x=\sum_{j=1}^{m}\varphi_j\otimes L_{\varphi_j}(x)$ which implies that (5) holds.

Since, as shown in Theorem 1, the point spectrum of N is given by $\{\lambda_\nu\}$ itself,

$$N=\sum_s{}'\lambda_s K_s+\int_\Delta z dK(z),$$

where $\sum_s{}'$ denotes the sum for all distinct eigenvalues λ_ν. In the same manner as above, we have therefore

$$Nx=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+\int_\Delta z dK(z)x$$

for every $x\in\mathfrak{H}$. Comparing this equality with (2), we obtain the desired relation (6).

The theorem has thus been proved.

Remark 1. Theorems 1 and 2 remain true, even if $\{\lambda_\nu\}$ is a finite set (inclusive of the multiplicity of each of its distinct elements). In that case, of course, $\{\varphi_\nu\}$ is also a finite set. Moreover, with very small modifications these theorems are valid, even if one of the orthonormal sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ is complete and hence the other is empty.

Remark 2. If the (one-dimensional or two-dimensional) measure of Δ is not zero, it can not be admitted that $\{\psi_\mu\}$ is a finite set: because the dimension of the orthogonal complement of the subspace determined by all eigenelements of N is never finite in that case.

Remark 3. Both the $\{\Psi_\mu\}$ and $\{\Psi_\mu^*\}$ are orthonormal sets orthogonal to $\{\varphi_\nu\}$.