6. On the Functional-Representations of Normal Operators in Hilbert Spaces

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Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ both be incomplete orthonormal sets in $\mathfrak{H}$ which have no element in common and together form a complete orthonormal set in that space; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed bounded sequence in the complex plane; let $\{u_{ij}\}$ be an infinite unitary matrix with $|u_{jj}|\neq 1$, $j=1,2,3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let L_x be the continuous linear functional associated with an arbitrary $x\in\mathfrak{H}$; and let $y\otimes L_x$ be the operator defined by $(y\otimes L_x)z=(z,x)y$ for an arbitrarily given $y\in\mathfrak{H}$ and for every $z\in\mathfrak{H}$. Then, with respect to the operator N defined as

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu},$$

where c is an arbitrarily given complex constant, I have proved in Vol. 37, No. 10 (1961) of Proceedings of the Japan Academy that not only the right-hand side converges uniformly, but that also N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$, and have defined the expression of the right-hand side as "the functional-representation of N".

The purpose of this paper is to prove that conversely every bounded normal operator N in $\mathfrak{H}$ is essentially expressible by such an infinite series of the continuous linear functionals associated with all the elements of a complete orthonormal set in $\mathfrak{H}$ as described above.

Theorem A. Let N be a bounded normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be its point spectrum (inclusive of the multiplicity of each eigenvalue of N); let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ be an orthonormal set determining the subspace $\mathfrak{M}$ determined by all the eigenelements of N, such that φ_ν is a normalized eigenelement corresponding to an arbitrary eigenvalue λ_ν of N; let $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ be an orthonormal set determining the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}$; and let L_f be the continuous linear functional associated with any $f\in\mathfrak{H}$. Then $\|N\psi_\mu\|^2$, $\mu=1,2,3,\cdots$, assume the same value, which will be denoted by σ; and if we choose arbitrarily a complex constant c with absolute value $\sqrt{\sigma}$ and put $\Psi_\mu=\sum_j u_{\mu j}\psi_j$, where $u_{\mu j}=(N\psi_\mu,\psi_j)/c$ and $\sum_j$

denotes the sum for all $\psi_j \in \{\psi_\mu\}$, the equality

$$(1) \qquad N = \sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu} + c \sum_\mu \Psi_\mu \otimes L_{\psi_\mu}$$

holds on the domain $\mathfrak{H}$ of N, and moreover the matrix (u_{kj}) associated with all the elements of $\{\psi_\mu\}$ is unitary and possesses the property $|u_{jj}| \neq 1$ for $j=1, 2, 3, \cdots$.

Proof. By hypotheses, $N\varphi_\kappa = \lambda_\kappa \varphi_\kappa$, $\kappa=1, 2, 3, \cdots$; and in addition, $\sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}(\varphi_\kappa) = \lambda_\kappa \varphi_\kappa$, $\kappa=1, 2, 3, \cdots$. Since, on the other hand, any element g of $\mathfrak{M}$ is expressed in the form $g = \sum_\nu (g, \varphi_\nu)\varphi_\nu$, these results permit us to assert that the equality $N = \sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}$ holds on $\mathfrak{M}$.

Now, let $\{K(z)\}$ be the complex spectral family associated with N, K_ν the eigenprojector corresponding to an arbitrary eigenvalue λ_ν of N, and $\Delta(N)$ the continuous spectrum of N. Then, since $N = \sum_\nu' \lambda_\nu K_\nu + \int_{\Delta(N)} z dK(z)$, where $\sum_\nu'$ denotes the sum for all distinct eigenvalues λ_ν in $\{\lambda_\nu\}$, and since $K_\nu \psi_\mu = 0$ for every pair of μ, ν, $N\psi_\mu$, $\mu=1, 2, 3, \cdots$, belong to $\mathfrak{N}$. Putting $\Phi_\mu = \sum_j C_{\mu j} \psi_j$ where $C_{\mu j} = (N\psi_\mu, \psi_j)$ and $\sum_j$ denotes the sum for all $\psi_j \in \{\psi_\mu\}$, we have therefore

$$\begin{aligned} \sum_\mu \Phi_\mu \otimes L_{\psi_\mu}(\psi_p) &= \Phi_p \\ &= \sum_j C_{pj} \psi_j \\ &= N\psi_p \end{aligned}$$

for every $\psi_p \in \{\psi_\mu\}$. This result leads us to the assertion that the equality $N = \sum_\mu \Phi_\mu \otimes L_{\psi_\mu}$ holds on $\mathfrak{N}$. Since, furthermore, any element $f \in \mathfrak{H}$ is uniquely expressed in the form $f = g + h$ where $g \in \mathfrak{M}$ and $h \in \mathfrak{N}$, and since

$$\sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}(f) = \sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}(g) = Ng$$

and

$$\sum_\mu \Phi_\mu \otimes L_{\psi_\mu}(f) = \sum_\mu \Phi_\mu \otimes L_{\psi_\mu}(h) = Nh,$$

we obtain

$$Nf = \sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu}(f) + \sum_\mu \Phi_\mu \otimes L_{\psi_\mu}(f),$$

which shows that the equality

$$(1') \qquad N = \sum_\nu \lambda_\nu \varphi_\nu \otimes L_{\varphi_\nu} + \sum_\mu \Phi_\mu \otimes L_{\psi_\mu}$$

holds on $\mathfrak{H}$.

If we next denote by δ an arbitrary subset with non-zero measure of $\Delta(N)$, $K(\delta)$ is a projector and hence the relation

$$\begin{aligned} (K(\delta)f, f') = {} & \left\| K(\delta)\frac{f+f'}{2} \right\|^2 - \left\| K(\delta)\frac{f-f'}{2} \right\|^2 \\ & + i\left[\left\| K(\delta)\frac{f+if'}{2} \right\|^2 - \left\| K(\delta)\frac{f-if'}{2} \right\|^2 \right] \end{aligned}$$

holds for every pair of $f, f' \in \mathfrak{H}$. Remembering that $K_\nu \psi_\mu = 0$ for every pair of μ, ν, we have therefore

$$
\begin{aligned}
(N\psi_\mu, N\psi_j) &= \int_{\Delta(N)} |z|^2 d(K(z)\psi_\mu, \psi_j) \\
&= \int_{\Delta(N)} |z|^2 d\Big[\Big\|K(z)\frac{\psi_\mu+\psi_j}{2}\Big\|^2 - \Big\|K(z)\frac{\psi_\mu-\psi_j}{2}\Big\|^2\Big] \\
&\quad + i\int_{\Delta(N)} |z|^2 d\Big[\Big\|K(z)\frac{\psi_\mu+i\psi_j}{2}\Big\|^2 - \Big\|K(z)\frac{\psi_\mu-i\psi_j}{2}\Big\|^2\Big].
\end{aligned} \tag{2}
$$

On the other hand, since

$$
\zeta(\delta) \equiv \Big\|K(\delta)\frac{\psi_\mu+\psi_j}{2}\Big\|^2 = \Big\|K(\delta)\frac{\psi_\mu-\psi_j}{2}\Big\|^2 + \Re(K(\delta)\psi_\mu, \psi_j) \geqq 0, \tag{3}
$$

the set function ζ defined here is an extended real valued and non-negative set function, defined on $\Delta(N)$ forming a (Boolean) ring, and such that $\zeta(0)=0$. Moreover the verification of the assertion that ζ is countably additive offers no difficulty. It is thus apparent that ζ is a measure. In consequence, by applying the mean value theorem for integrals to the equality

$$
\int_{\Delta(N)} |z|^2 d\Big\|K(z)\frac{\psi_\mu+\psi_j}{2}\Big\|^2 = \int_{\Delta(N)} |z|^2 d\Big[\Big\|K(z)\frac{\psi_\mu-\psi_j}{2}\Big\|^2 + \Re(K(z)\psi_\mu, \psi_j)\Big]
$$

deduced from (3), we find from the boundedness of N that

$$
\begin{aligned}
\rho\int_{\Delta(N)} d\Big\|K(z)\frac{\psi_\mu+\psi_j}{2}\Big\|^2 &= \rho\int_{\Delta(N)} d\Big[\Big\|K(z)\frac{\psi_\mu-\psi_j}{2}\Big\|^2 + \Re(K(z)\psi_\mu, \psi_j)\Big] \\
&= \rho\int_{\Delta(N)} d\Big\|K(z)\frac{\psi_\mu-\psi_j}{2}\Big\|^2 + \rho\int_{\Delta(N)} d[\Re(K(z)\psi_\mu, \psi_j)],
\end{aligned}
$$

where ρ is a suitable positive constant such that $\inf_{z\in\Delta(N)} |z|^2 \leqq \rho \leqq \sup_{z\in\Delta(N)} |z|^2 \leqq \|N\|^2$. As will be found from (2), this result shows that

$$
\begin{aligned}
\Re(N\psi_\mu, N\psi_j) &= \rho\int_{\Delta(N)} d[\Re(K(z)\psi_\mu, \psi_j)] \\
&= \rho\Re(K(\Delta(N))\psi_\mu, \psi_j) \\
&= \rho\Re(\psi_\mu, \psi_j) \\
&= 0
\end{aligned}
$$

for every pair of two distinct elements $\psi_\mu, \psi_j \in \{\psi_\mu\}$.

In the same manner as above, we find that $\Im(N\psi_\mu, N\psi_j)=0$ for all distinct $\psi_\mu, \psi_j \in \{\psi_\mu\}$. Consequently the relation $(N\psi_\mu, N\psi_j)=0$ holds for every pair of distinct $\psi_\mu, \psi_j \in \{\psi_\mu\}$.

Furthermore, by reasoning exactly like that applied to $\Re(N\psi_\mu, N\psi_j)$, we can find that

$$
\begin{aligned}
&\Re(N(\psi_\mu+\psi_j), N(\psi_\mu-\psi_j)) \\
&\quad = \gamma\int_{\Delta(N)} d[\Re(K(z)(\psi_\mu+\psi_j), \psi_\mu-\psi_j)], \quad (0<\gamma\leqq\|N\|^2), \\
&\quad = \gamma\Re(\psi_\mu+\psi_j, \psi_\mu-\psi_j) \\
&\quad = 0,
\end{aligned}
$$

whereas $(N(\psi_\mu+\psi_j), N(\psi_\mu-\psi_j))=||N\psi_\mu||^2-||N\psi_j||^2$. Hence all the $||N\psi_\mu||^2$ for $\mu=1, 2, 3,\cdots$ assume the same value, which will be denoted by σ.

We now choose arbitrarily a complex number c such that $|c|^2=\sigma$ and put $\Psi_\mu=\sum_j u_{\mu j}\psi_j$ where $u_{\mu j}=C_{\mu j}/c$. Then, by making use of the just established relations

$$(N\psi_\mu, N\psi_p)=\begin{cases}|c|^2 & (\mu=p)\\ 0 & (\mu\neq p)\end{cases}, \quad \mu, p=1, 2, 3,\cdots,$$

and of the fact that $N\psi_\mu$ belongs to $\mathfrak{N}$ for every $\psi_\mu\in\{\psi_\mu\}$, we have

$$\begin{aligned}(\Psi_\mu, \Psi_p)&=\sum_j u_{\mu j}\bar{u}_{pj}\\ &=\sum_j(N\psi_\mu, \psi_j)\overline{(N\psi_p, \psi_j)}/|c|^2\\ &=(N\psi_\mu, N\psi_p)/|c|^2\\ &=\begin{cases}1 & (\mu=p)\\ 0 & (\mu\neq p)\end{cases}.\end{aligned}$$

In addition, it is clear that (1′) is expressed in the form (1).

Thus it remains only to prove that

$$(4)\qquad \sum_j u_{j\mu}\bar{u}_{jp}=\begin{cases}1 & (\mu=p)\\ 0 & (\mu\neq p)\end{cases}$$

and that $|u_{jj}|\neq 1$ for $j=1, 2, 3,\cdots$.

To prove the validity of these relations, we consider the adjoint operator N^* of N. Then we have $N^*\varphi_\nu=\bar{\lambda}_\nu\varphi_\nu$, $N^*=\int_G \bar{z}dK(z)$ where G denotes the complex z-plane, and $(N^*\psi_\mu, \psi_j)=\overline{(N\psi_j, \psi_\mu)}=\overline{C}_{j\mu}$. Accordingly, by the same reasoning as that used to establish the functional-representation (1) of N it can be verified without difficulty that

$$N^*=\sum_\nu\bar{\lambda}_\nu\varphi_\nu\otimes L\varphi_\nu+\bar{c}\sum_\mu\Psi_\mu^*\otimes L_{\psi_\mu},$$

where $\Psi_\mu^*=\sum_j\bar{u}_{j\mu}\psi_j$, and that (4) is valid. Thus the matrix (u_{kj}) associated with all the elements of $\{\psi_\mu\}$ is unitary. Furthermore it is seen that

$$(5)\qquad |u_{jj}|=|C_{jj}|/|c|=|(N\psi_j, \psi_j)|/||N\psi_j||,\ j=1, 2, 3,\cdots,$$

and that $||N\psi_j||^2=\sum_\mu|(N\psi_j, \psi_\mu)|^2$ in accordance with the Parseval identity and the fact that $N\psi_j$ belongs to $\mathfrak{N}$. On the other hand, it never occurs that $(N\psi_j, \psi_\mu)$ vanishes for every ψ_μ different from ψ_j; for otherwise ψ_j would become an eigenelement of N, contrary to hypotheses. Hence $||N\psi_j||>|(N\psi_j, \psi_j)|$. By virtue of the application of this inequality to (5), we obtain $|u_{jj}|<1$ for $j=1, 2, 3,\cdots$.

With these results, the proof of the theorem is complete.

Remark 1. Since it is easily verified by means of (4) that

$$\sum_\mu(h, \Psi_\mu)\Psi_\mu=\sum_\mu(h, \psi_\mu)\psi_\mu=h$$

for every $h\in\mathfrak{N}$, the set $\{\Psi_\mu\}$ associated with $\{\psi_\mu\}$ is an orthonormal set determining $\mathfrak{N}$; and moreover it is seen that the same result is true of $\{\Psi_\mu^*\}$.

Remark 2. It is found immediately from the method of the proof of Theorem A that, if the (one-dimensional or two-dimensional) measure of $\Delta(N)$ is zero, the second member in the right-hand side of (1) vanishes and $\{\varphi_\nu\}$ is a complete orthonormal set, and that, if, on the contrary, the point spectrum of N is empty, N is expressd by that second member in which the orthonormal set $\{\psi_\mu\}$ is complete.

Corollary A. If, in Theorem A, $f(z)$ is a function holomorphic on the closed domain $D\{z: |z|\leqq\|N\|\}$, then $\|f(N)\psi_\mu\|^2$, $\mu=1, 2, 3, \cdots$, assume the same value, which will be denoted by σ'; and if, in addition, we choose arbitrarily a complex constant c' with absolute value $\sqrt{\sigma'}$ and put $\Psi'_\mu=\sum_j u'_{\mu j}\psi_j$ where $u'_{\mu j}=(f(N)\psi_\mu, \psi_j)/c'$ and $\sum_j$ denotes the sum for all $\psi_j\in\{\psi_\mu\}$, then the equality

$$f(N)=\sum_\nu f(\lambda_\nu)\varphi_\nu\otimes L_{\varphi_\nu}+c'\sum_\mu \Psi'_\mu\otimes L_{\psi_\mu}$$

holds on $\mathfrak{H}$ and the matrix (u'_{kj}) associated with all the elements of $\{\psi_\mu\}$ possesses the same characters as those of the matrix (u_{kj}) described in Theorem A.

Proof. Since, by definition, we have $f(N)=\int_D f(z)dK(z)$, which implies that the adjoint operator $f^*(N)$ of $f(N)$ is given by $f^*(N)=\int_D \overline{f(z)}dK(z)$, and since, by hypotheses, $f(z)$ is holomorphic on D, there is no difficulty in showing that

1° $f(N)$ is a bounded normal operator in $\mathfrak{H}$;

2° the point spectrum of N is given by $\{f(\lambda_\nu)\}_{\nu=1,2,3,\ldots}$, and φ_ν is an eigenelement of $f(N)$ corresponding to the eigenvalue $f(\lambda_\nu)$;

3° the continuous spectrum of $f(N)$ also is given by the image of $\Delta(N)$ by $f(z)$.

Accordingly the present corollary is a direct consequence of Theorem A.

Correction to Sakuji Inoue: "Functional-Representations of Normal Operators in Hilbert Spaces and Their Applications" (Proc. Japan Acad., Vol. 37, No. 10, 614–618 (1961)).

Page 614, line 17 from bottom: read "$\sum_{\nu=1}^{\infty}$" in place of "$\sum_{j=1}^{\infty}$".

Page 615, line 1: read "b_μ" in place of "$b\mu$".

Page 616, line 1: read "$\overline{L_{\varphi_\nu}(y)}$ and $\overline{L_{\psi_\kappa}(y)}$" in place of "$\overline{L_{\psi_\kappa}(y)}$ and $\overline{L_{\varphi_\nu}(y)}$".

Page 617, line 18: read "relations" in place of "velations".