5. Rotationally Invariant Measures in the Dual Space of a Nuclear Space

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The purpose of the present paper is to show that any rotationally invariant measure in a Hilbert space (more exactly, in the dual space of a nuclear space) is expressed as a superposition of Gaussian measures. The author intends to discuss this problem in details in another paper. So, we shall indicate the proof only briefly.

§ 1. **Preliminaries.** We shall explain our problem more exactly. Let L be a real topological vector space which is defined by countable Hilbertian norms and is nuclear. Let L^* be the dual space of L. For L and L^*, R. A. Minlos proved the following generalization of Bochner's theorem:

For every continuous and positive definite function $\chi(\xi)$ on L there exists a uniquely determined Borel measure μ on L^*, which fulfils the relation

$$\chi(\xi)=\int \exp [i(\xi, x)]d\mu(x). \qquad (1)$$

Conversely, for any μ the relation (1) defines a continuous and positive definite function $\chi(\xi)$, the characteristic function of μ.

Now let H be the completion of L by a continuous Hilbertian norm $\|\cdot\|$. Then we may suppose $L\subset H\subset L^*$.

We shall call an orthogonal operator u on H a *rotation* of L, if it satisfies the following conditions:

1) u maps L onto L;

2) u is homeomorphic on L.

All the rotations of L form a group, which we shall call the *rotation group* of L and denote by $O(L)$. If we identify u and u^{-1*}, $O(L)$ can be regarded a transformation group of L^* onto itself.

Now let G be any group of homeomorphic transformations of L^* onto itself. From a given measure μ on L^*, we define the transformed measure $\tau_g\mu$ as follows:

$$\tau_g\mu(A)=\mu(gA), \text{ for any Borel set } A.$$

If $\mu=\tau_g\mu$ for any $g\in G$, then μ is called *G-invariant*. It $\tau_g\mu$ is absolutely continuous with respect to μ for any $g\in G$, then μ is called *G-quasi-invariant*. Finally, μ is called *G-ergodic*, if μ is G-quasi-invariant and the condition $A=gA$ (for all $g\in G$) implies $A=\phi$ or $A=L^*$ modulo nullsets. In the case of $G=O(L)$, we simply call μ O-invariant or O-ergodic.

Since $L \subset L^*$, the translations by an element of L can be defined in L^*: $x \to x+\xi$. All such translations form a group, which we identify with L. Hence, we can define the concept of L-quasi-invariance or L-ergodicity.

It is easy to show that the function

$$\chi(\xi)=\exp\left[-\frac{c^2}{2}\|\xi\|^2\right] \quad (c>0) \tag{2}$$

is continuous and positive definite on L. The corresponding measure μ_c on L^* is called *Gaussian measure* with variance c^2. It can be shown that μ_c is L-ergodic, O-ergodic and O-invariant.

Now consider a complete orthonormal base $\{\xi_k\}$ in L, and define the function $f(x)$ on L^* as follows:

$$f(x)=\overline{\lim_k}\frac{|(\xi_{k,x})|}{\sqrt{2\log k}}. \tag{3}$$

Then, we have $\mu_c(f^{-1}(c))=1$ $(=\mu_c(L^*))$. Especially we see that if $c\neq c'$, μ_c is singular with respect to $\mu_{c'}$.

§2. **Main Results.** Our main object is the characterization of an O-invariant measure as a superposition of Gaussian measures.

THEOREM 1. *A measure μ on L^* is O-invariant if and only if there exist a real number $\alpha\geqq 0$ and a summable measure $m(c)$ on the interval $(0, \infty)$ such that for any Borel set A in L^*,*

$$\mu(A)=\int_{0<c<\infty}\mu_c(A)dm(c)+\alpha\delta(A) \tag{4}$$

where δ denotes the Dirac measure on the origin of L^*.

Proof of sufficiency. Since both μ_c and δ are O-invariant, any measure of the form of (4) is evidently O-invariant.

To prove the converse, we need some lemmas.

LEMMA 1. *A measure μ on L^* is O-invariant if and only if the characteristic function $\chi(\xi)$ depends only on $\|\xi\|$.*

LEMMA 2 (*Bernstein's theorem*). *If a function $\varphi(t)$ defined on $[0, \infty)$ is completely monotonic and right continuous at $t=0$, then there exists a summable measure $\tilde{m}(s)$ on $[0, \infty)$ such that*

$$\varphi(t)=\int_{0\leqq s<\infty}\exp(-st)d\tilde{m}(s).$$

Here, a function $\varphi(t)$ is called completely monotonic if

$$(-1)^n\Delta_\alpha^{(n)}\varphi(t)\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}\varphi(k\alpha+t)\geqq 0$$

for $n=0,1,2,\cdots$, $t\geqq 0$, $\alpha\geqq 0$.

LEMMA 3. *If $\varphi(\|\xi\|^2)$ is positive definite on L, then*

a) $\varphi(\|\xi\|^2)\geqq 0$;

b) $\Delta_\alpha^{(1)}\varphi(\|\xi\|^2)\equiv\varphi(\|\xi\|^2+\alpha)-\varphi(\|\xi\|^2)$ *is negative definite.*

REMARK. For the validity of Lemma 3, it is essential that L is infinite dimensional.

Now we shall sketch the *proof of necessity* of Theorem 1. If μ is O-invariant, then the characteristic function $\chi(\xi)$ depends only on $||\xi||$ (Lemma 1), so that we can write $\chi(\xi)=\varphi(||\xi||^2)$. Then by Lemma 3 we can show that $\varphi(t)$ is completely monotonic. Hence by Lemma 2, there exists a measure $\widetilde{m}(s)$ on $[0, \infty)$ such that

$$\chi(\xi)=\int_{0\leqq s<\infty} \exp(-s||\xi||^2)d\widetilde{m}(s).$$

Putting $s=c^2/2$, we carry out the integral with regard to c. Then there exists a measure $m(c)$ on $[0, \infty)$ such that

$$\chi(\xi)=\int_{0\leqq c<\infty} \exp\Big(-\frac{c^2}{2}||\xi||^2\Big)dm(c).$$

Therefore, we get the equality (4) where $\alpha=m(\{0\})$, in virtue of the correspondence of measures and characteristic functions. (q. e. d.)

§ 3. **Ergodicity.** Finally we discuss the ergodicity of measures.

THEOREM 2. *Let μ be an O-invariant measure on L^*.*

a) *If μ is L-quasi-invariant, then there exists a summable measure $m(c)$ on $(0, \infty)$ such that*

$$\mu(A)=\int_{0<c<\infty} \mu_c(A)dm(c). \tag{5}$$

b) *If μ is L-ergodic, then $\mu=\mu_c$ for some $c>0$.*

c) *If μ is O-ergodic, then $\mu=\mu_c$ for some $c>0$ or $\mu=\delta$.*

PROOF. a) is evident, for $\delta(A)$ is not L-quasi-invariant.

To prove b), we use the function $f(x)$ which we have defined by the formula (3) in § 1. It is easy to show that $f(x)=f(x+\xi)$ for all $x\in L^*$ and $\xi\in L$. Hence, for any given $c'>0$, $A_{c'}\equiv f^{-1}((0, c'))$ is an L-invariant set. Therefore, if μ is L-ergodic, then $\mu(A_{c'})=0$ or $\mu(A_{c'})=1$.

On the other hand,

$$\mu_c(A_{c'})=\begin{cases} 1 & (\text{if } c<c') \\ 0 & (\text{if } c\geqq c') \end{cases}$$

so that from (5), we have

$$\mu(A_{c'})=\int_{0<c<c'} dm(c)=m((0, c'))=0 \text{ or } 1.$$

This equality holds for any $c'>0$, thus $m(c)=\delta(c-c_0)$ for some c_0. Hence, again from (5), we get $\mu=\mu_{c_0}$.

With some modifications, c) is proved in a similar way. (q. e. d.)

COROLLARY. *For O-invariant measure (except the Dirac measure), L-ergodicity is equivalent with O-ergodicity.*