# 4. Decomposition of Representations of the Three-Dimensional Lorentz Group

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The purpose of the present paper is the explicit description of decomposition of the unitary representations of the three-dimensional Lorentz group, which are constructed on factor spaces, into irreducible representations. We solve this problem by infinitesimal method. This problem arises as one step of decomposing tensor products of irreducible representations of the inhomogeneous Lorentz group into irreducible ones. The proof of the results of the present paper and the details about the last problem will be published in other papers.

§ 1. In order to include the so-called spinor representations, we consider the real special linear group $G$ of second order. $G$ is the two-fold covering group of the three-dimensional Lorentz group. Now we shall give some definitions which are necessary to describe our problem and results exactly.

$G$ is generated by three subgroups of the following types:

$$S=\left\{s(\theta)=\begin{pmatrix}\cos(\theta/2), & -\sin(\theta/2)\\ \sin(\theta/2), & \cos(\theta/2)\end{pmatrix};\ -2\pi<\theta\leq 2\pi\right\},$$

$$D=\left\{d^{\pm}(\zeta)=\pm\begin{pmatrix}\exp(\zeta/2), & 0\\ 0, & \exp(\zeta/2)\end{pmatrix};\ -\infty<\zeta<\infty\right\},$$

$$L=\left\{l^{\pm}(t)=\pm\begin{pmatrix}\mathrm{ch}(t/2), & \mathrm{sh}(t/2)\\ \mathrm{sh}(t/2), & \mathrm{ch}(t/2)\end{pmatrix};\ -\infty<t<\infty\right\}.$$

For arbitrary unitary representation $\mathfrak{R}$ of $G$, denote the corresponding operators to the generators of Lie algebra with respect to these parameters by $H(S,\mathfrak{R})$, $H(D,\mathfrak{R})$, $H(L,\mathfrak{R})$ respectively.

The factor space $S\backslash G$ can be imbedded in $G$ by the correspondence of coset to its representative $e$ or $d^{+}(\zeta)s(\theta)$ $(0<\zeta<\infty,\ -\pi<\theta\leq\pi)$. In the same way, for the case of $D\backslash G$, the element $e$ or $l^{+}(t)s(\theta)$ $(-\infty<t<\infty,\ -\pi<\theta\leq\pi)$ are representatives.

§ 2. The representations to be decomposed are the so-called induced representations of $G$ from one-dimensional representations of $S$ or $D$, that is, $\exp(ik\theta)$ ($k$: integer or half-integer) of $S$, and $\exp(i\tau\zeta)$ ($\tau$; real) of $D$, or $\pm\exp(i\tau\zeta)$ for spinor case. The spaces of these representations are $L^2_\mu$ by the $G$-invariant measure $\mu$ over the factor space $\Omega=S\backslash G$ or $D\backslash G$ respectively, and the operator $U_g$ of representation is defined by a function $\alpha(\omega,g)$ over $\Omega\times G$ for any element $f(\omega)$ of $L^2_\mu$: $(U_gf)(\omega)\equiv\alpha(\omega,g)f(\omega\cdot g)$. The multiplier $\alpha(\omega,g)$ is given as follows:

In the case of $\Omega=S\backslash G$: put

$$\alpha(e, g)\equiv\exp(ik\theta), \quad \text{where } g=s(\theta)\cdot\omega.$$

In the case of $\Omega=D\backslash G$: put

$$\alpha(e, g)\equiv\exp(i\tau\zeta), \quad \text{for non-spinor case,}$$
$$\equiv\pm\exp(i\tau\zeta), \text{ for spinor case,}$$

where $g=d^{\pm}(\zeta)\omega$. And finally put

$$\alpha(\omega, g)\equiv\alpha(e, \omega\cdot g).$$

Let us denote the representations constructed above by $\mathfrak{R}(k)$, $\mathfrak{R}_1(\tau)$, $\mathfrak{R}_2(\tau)$ respectively.

§ 3. Here we quote the well-known result of V. Bargmann about irreducible unitary representations of $G$. They are classified and denoted as follows:

(a) Principal series (non-spinor) $C_l^0$ $(1/4\leq l<\infty)$.
(a′) Principal series (spinor) $C_l^{1/2}$ $(1/4<l<\infty)$.
(b) Supplementary series $C_l^0$ $(0<l<1/4)$.
(c) Discrete series (non-spinor) $D_n$ ($n$: integer, $\neq 0$).
(c′) Discrete series (spinor) $D_n$ ($n$: half-integer).
(d) Identity representation $I$ (identity operator for all $g$).

The following results are valid:

(1°) In the case of $\mathfrak{R}=C_l^0$ or $C_l^{1/2}$,

$$\Delta(\mathfrak{R})\equiv(H(D,\mathfrak{R}))^2+(H(L,\mathfrak{R}))^2-(H(S,\mathfrak{R}))^2=-l\cdot I.$$

(2°) The representation of $S$, which is the restriction of $C_l^0$, $C_l^{1/2}$, or $D_n$ to $S$, can be written as a direct sum $\sum\oplus\rho(k)$ of the representations $\rho(k)\equiv\exp(ik\theta)$ of $S$, in which multiplicity is one for every $k$, and the summation by $k$ runs over the set of

i) all integers for the representations (a), (b),
ii) all half-integers for (a′),
iii) integers which are between $n$ and (sign $n)\infty$ for (c),
iv) half-integers which are between $n$ and (sign $n)\infty$ for (c′).

Moreover in these cases the operator $F^{\pm}(\mathfrak{R})\equiv H(D,\mathfrak{R})\mp iH(L,\mathfrak{R})$ gives a mapping of the subspace corresponding to $\rho(k)$ onto the space corresponding to $\rho(k\pm 1)$, when the former is not trivial.

§ 4. Using the above definitions and notations our results are formulated as follows:

1) For any integer $k$,

$$\mathfrak{R}(k)=\sum_n\oplus D_n\oplus\int_{1/4}^{\infty}\oplus C_l^0 dl,$$

where $n$ are integers and

for $k\geq 1, k\geq n\geq 1$,
for $k=0$, the first summand disappears,
for $k\leq -1, k\leq n\leq -1$.

2) For half-integer $k$,

$$\mathfrak{R}(k)=\sum_{n}\oplus D_n\oplus\int_{1/4}^{\infty}\oplus C_l^{1/2}dl,$$

where $n$ are half-integers and

for $k\geq 3/2,\ k\geq n\geq 3/2$,

for $k=\pm 1/2$, the first summand disappears,

for $k\leq -3/2,\ k\leq n\leq -3/2$.

3) $$\mathfrak{R}_1(\tau)=\sum_{n}\oplus D_n\oplus[2]\int_{1/4}^{\infty}\oplus C_l^0 dl,$$

where $n$ runs over all integers except 0, and the number [2] means that the multiplicity is two.

4) $$\mathfrak{R}_2(\tau)=\sum_{n}\oplus D_n\oplus[2]\int_{1/4}^{\infty}\oplus C_l^{1/2}dl,$$

where $n$ runs over all half-integers.

We remark that in these decompositions, representations of the supplementary series and the identity representation do not appear.

As for the continuous direct sums of these decompositions, we can calculate the weight functions for fixed sets of eigenvectors. In the non-spinor cases 1) and 3) they have very simple forms as follows, although in the spinor cases 2) and 4) they are limits of hypergeometric functions, and have complicated forms.

The case 1). The eigenvectors in the space corresponding to $\rho(0)$ are of the form of $\mathfrak{P}^{|k|}_{-(1/2)-i\sigma}(\mathrm{ch}\,\xi)$, and the weight function is

$$(\pi/2)(\mathrm{th}\,\sigma\pi)\{((1/2)^2+\sigma^2)((3/2)^2+\sigma^2)\cdots((|k|-1/2)^2+\sigma^2)\}^{-1},$$

where $\sigma=(l-(1/4))^{1/2}$.

The case 3). There are two independent families of eigenvectors, one of which is generated by $\mathfrak{P}^{i|\tau|}_{-(1/2)-i\sigma}(i\,\mathrm{sh}\,t)$ and the other is generated by $\mathfrak{Q}^{i|\tau|}_{-(1/2)-i\sigma}(i\,\mathrm{sh}\,t)$, and the weight functions are

$(2\pi)\,\mathrm{sh}(\sigma\cdot\pi)\,|\Gamma(z^+)\Gamma(z^-)|^{-2}((\mathrm{ch}(\tau\cdot\pi))^2+\mathrm{sh}(\sigma\cdot\pi))^2)^{-1}$ for the former,

$(2\pi)^{-3}\,\mathrm{sh}(\sigma\cdot\pi)\,|\Gamma(z^+)\Gamma(z^-)|^2$ for the latter,

where $\sigma=(l-(1/4))^{1/2},\ z^{\pm}=(1/4)+i((\sigma\pm|\tau|)/2)$.

§ 5. Lastly we shall mention briefly about our principle of the irreducible decomposition. For a unitary representation $\mathfrak{R}$ of $G$, by the property (2°) of § 2, we can separate the generating vector belonging to $D_k$-component, by means of calculating the kernel of $F^{\pm}(\mathfrak{R})$ in the subspace corresponding to $\rho(k)$ for the restriction of $\mathfrak{R}$ to $S$. Next, irreducible components which are equivalent to $I$, $C_l^0$ or $C_l^{1/2}$ can be obtained by eigenfunction expansion with respect to $\Delta(\mathfrak{R})$ in the space corresponding to $\rho(0)$ or $\rho(1/2)$ for the restriction of $\mathfrak{R}$ to $S$.