23. An Asymptotic Property of a Gap Sequence

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1. **Introduction.** Let $f(t)$ be a real measurable function satisfying

$$(1.1)\qquad f(t+1)=f(t),\ \int_0^1 f(t)dt=0 \text{ and } \int_0^1 f^2(t)dt<+\infty,$$

and $\{n_k\}$ be a lacunary sequence of positive integers, that is,

$$(1.2)\qquad n_{k+1}/n_k>q>1.$$

Then the sequence of functions $\{f(n_kt)\}$, although themselves not independent, exibits the properties of independent random variables (c.f. [3]). In [2] Professor S. Izumi proved that if $f(t)$ satisfies certain smoothness conditions, then $\{f(2^kt)\}$ obeys the law of the iterated logarithm. However if we put $f(t)=\cos 2\pi t+\cos 4\pi t$ and $n_k=2^k-1$, then, by the theorem of Erdös and Gál [1], we have,

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{N\log\log N}}\sum_{k=1}^{N}f(n_kt)=2\cos\pi t,\quad \text{a.e. in } t.$$

This shows that $\{f(n_kt)\}$ does not necessarily obey the law of the iterated logarithm even if $f(t)$ is a trigonometric polynomal.

In §§ 2–4 we shall prove the following

Theorem. Let $f(t)$ and $\{n_k\}$ satisfy (1.1) *and* (1.2) *respectively and $f(t)$ be a function of* Lip α, $0<\alpha\leq 1$. *Then we have,*

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{N\log\log N}}\sum_{k=1}^{N}f(n_kt)\leq C,\quad \text{a.e. in } t,$$

where C is a positive constant depending on $f(t)$ and q in (1.1).

2. **Preliminary.** From now on let $f(t)$ and $\{n_k\}$ satisfy the conditions of the theorem. For simplicity of writing we may assume that

$$f(t)\sim\sum_{k=1}^{\infty}c_k\cos 2\pi kt.$$

The proof is the same in the general cases as we can see by writing

$$a_k\cos 2\pi kt+b_k\sin 2\pi kt=\rho_k\cos 2\pi k(t-\xi_k).$$

In this paragraph let N be any fixed integer satisfying

$$(2.1)\qquad q^N>3N^\beta$$

where β is a positive constant such that $\alpha\beta=6$.
Let us put, for $m=0,1,\cdots,$

$$(2.2)\qquad g(t)=\sum_{k=1}^{N^\beta}c_k\cos 2\pi kt \text{ and } U_m(t)=\sum_{l=Nm+1}^{N(m+1)}g(n_lt).$$

Since $f(t)\in$ Lip α and $\alpha\beta=6$, we have for some constant A,

$$(2.3)\qquad |f(t)-g(t)|<AN^{-\alpha\beta}\log N\leq AN^{-6}\log N,\quad \text{for all } t,$$

and

$$\sum_{k=n}^{\infty} c_k^2 \leq A n^{-2\alpha}. \tag{2.3'}$$

Further for simplicity of writing we may assume, by (2.3), that

$$|g(t)| \leq 1. \tag{2.4}$$

Lemma 1. *If λ is a positive number satisfying*

$$2\lambda N^2 < 1, \tag{2.5}$$

then we have, for any positive integer k,

$$I(\lambda, k) = \int_0^1 \exp\{\lambda \sum_{m=0}^{k-1} U_{2m}(t)\} dt \leq e^{\lambda^2 BNk}, \tag{2.5'}$$

and

$$I'(\lambda, k) = \int_0^1 \exp\{\lambda \sum_{m=1}^{k} U_{2m-1}(t)\} dt \leq e^{\lambda^2 BNk}, \tag{2.5''}$$

where B is a positive constant depending only on $f(t)$ and q.

Proof. If $|z| < 1/2$, then it is easily seen that

$$e^z < (1 + z + z^2/2) e^{2|z|^3}.$$

By (2.2), (2.4) and (2.5), we have

$$|2\lambda U_{2m}(t)| \leq 2\lambda N < N^{-1},$$

and

$$2\sum_{m=0}^{k-1} |\lambda U_{2m}(t)|^3 \leq 2\sum_{m=0}^{k-1} \lambda^3 N^3 < \lambda^2 Nk.$$

From the above relations and (2.5'), we obtain

$$I(\lambda, k) < e^{\lambda^2 Nk} \int_0^1 \prod_{m=0}^{k-1} \{1 + \lambda U_{2m}(t) + \lambda^2 U_{2m}^2(t)/2\} dt. \tag{2.6}$$

On the other hand by (2.2), we have

$$U_m^2(t) = \sum_{l=Nm+1}^{N(m+1)} g^2(n_l t) + 2 \sum_{l=Nm+1}^{N(m+1)-1} \sum_{j=l+1}^{N(m+1)} g(n_l t) g(n_j t)$$

and, for $Nm < l < j \leq N(m+1)$,

$$g^2(n_l t) - \frac{1}{2} \sum_{k=1}^{N^\beta} c_k^2$$

and

$$g(n_l t) g(n_j t) - \frac{1}{2} \sum_{\substack{0 < r, s \leq N^\beta \\ |n_j r - n_l s| < n_{Nm}}} c_r c_s \cos 2\pi (n_l s - n_j r) t$$

are both sums of trigonometric functions whose frequencies lie between n_{Nm} and $N^\beta (n_j + n_l)$. Hence if we define $W_m(t)$ as follows:

$$U_m^2(t) = W_m(t) + \frac{N}{2} \sum_{k=1}^{N^\beta} c_k^2 + \sum_{l=Nm+1}^{N(m+1)-1} \sum_{j=l+1}^{N(m+1)} \sum_{\substack{0 < r, s \leq N^\beta \\ |u_j r - n_l s| < n_{Nm}}} c_r c_s \cos 2\pi (n_l s - n_j r) t. \tag{2.7}$$

Then $W_m(t)$ is the sum of trigonometric functions whose frequencies lie between n_{Nm} and $2N^\beta n_{N(m+1)}$. If $V_m(t)$ denotes the last term of the right hand side of (2.7), then we have, by (1.1) and (2.3),

$$\begin{aligned} |V_m(t)| &\leq \sum_{l=Nm+1}^{N(m+1)-1} \sum_{j=l+1}^{N(m+1)} \sum_{\substack{0 < r, s \leq N^\beta \\ |n_l s - n_j r| < n_{Nm}}} |c_r c_s| \leq \sum_{l=Nm+1}^{N(m+1)-1} \sum_{j>l} \sum_{|s - r n_j / n_l| < 1} |c_r c_s| \\ &\leq \sum_{l=Nm}^{N(m+1)} \sum_{j>l} \{\sum_{r=1}^{\infty} c_r^2\}^{1/2} \{\sum_{s > n_j / n_l^{-1}} c_s^2\}^{1/2} \leq A \sum_{l=Nm}^{N(m+1)} \sum_{j>l} q^{-\alpha|j-l|} \leq NB_1, \end{aligned} \tag{2.8}$$

where B_1 is a positive constant depending only on $f(t)$ and q. Putting $B_2=\frac{1}{2}\sum_{k=1}^{\infty}c_k^2+B_1$, it follows, from (2.7) and (2.8), that

$$U_m^2(t)\leq NB_2+W_m(t).$$

Hence we have, by (2.6),

$$I(\lambda, k)\leq e^{\lambda^2 Nk}\int_0^1\prod_{m=0}^{k-1}\{1+\lambda^2 NB_2/2+\lambda U_{2m}(t)+\lambda^2 W_{2m}(t)/2\}dt. \tag{2.9}$$

If $d_{2m}\cos 2\pi u_{2m}t$ is any term of the trigonometric polynomial $\lambda U_{2m}(t)+\lambda^2 W_{2m}(t)/2$, then by (2.2) and the above discussions, we have $n_{2Nm}\leq u_{2m}\leq 2N^{\beta}n_{(2m+1)N}$. Therefore we have, by (1.1) and (2.1),

$$u_{2m}-\sum_{k=0}^{m-1}u_{2k}\geq n_{2mN}-2N^{\beta}\sum_{k=0}^{m-1}n_{N(2k+1)}>n_{2mN}(1-2N^{\beta}\sum_{k=0}^{m-1}q^{-N(2k+1)})>0.$$

This implies that for any $(k_0, k_1,\cdots, k_l)$ such that $0\leq k_0<k_1\cdots k_l<k$

$$\int_0^1\prod_{i=0}^{l}\cos 2\pi u_{2ki}t\,dt=0.$$

Hence we have

$$\int_0^1\prod_{m=0}^{k-1}\{1+\lambda^2 NB_2/2+\lambda U_{2m}(t)+\lambda^2 W_{2m}(t)/2\}dt=(1+\lambda^2 B_2 N/2)^k\leq e^{\lambda^2 B_2 Nk/2}.$$

Putting $B=1+B_2/2$, we can prove (2.5) by (2.9) and the above relation. In the same way we can prove (2.5$''$).

3. Fundamental Inequality. Using Lemma 1 we prove

Lemma 2. *There exist positive constants B_0 and M_0 depending only on $f(t)$ and q such that if M and positive λ satisfy the conditions*

$$M>M_0 \text{ and } 4\lambda M^{1/3}<1, \tag{3.1}$$

then we have

$$J(\lambda, M)=\int_0^1\exp\{\lambda\sum_{k=1}^{M}f(n_k t)\}dt\leq 2e^{\lambda^2 B_0 M}. \tag{3.2}$$

Proof. Let N be a positive integer such that $N^6\leq M<(N+1)^6$. Then if $M>M_1$ for some M_1, N satisfies (2.1). For this N construct the functions $g(t)$ and $U_m(t)$ by means of (2.2). Then from (2.3) and (3.1) we obtain, if $M>M_2$ for some M_2,

$$\lambda\sum_{l=1}^{M}|f(n_l t)-g(n_l t)|\leq A\lambda MN^{-6}\log N<\frac{1}{2}\log 2.$$

Next let k be a positive integer such that $N(2k+1)\leq M<N(2k+3)$. Then we have, by (2.2), (2.4) and (3.1), for $M>M_3$

$$\lambda|\sum_{l=1}^{M}g(n_l t)-\sum_{m=0}^{2k}U_m(t)|\leq\lambda\sum_{l=(2k+1)N+1}^{M}|g(n_l t)|<2\lambda N<\frac{1}{2}\log 2.$$

Therefore if $M>M_0=\max(M_1, M_2, M_3)$, we have by the above relations and (3.1)

$$J(\lambda, M)<2\int_0^1\exp\{\lambda\sum_{m=0}^{2k}U_m(t)\}dt,$$

and, by the Schwarz inequality,

$$J(\lambda, M)<2\Big[\int_0^1 \exp\{2\lambda\sum_{m=0}^{k}U_{2m}(t)\}\,dt\int_0^1 \exp\{2\lambda\sum_{m=1}^{k}U_{2m-1}(t)\}dt\Big]^{1/2}.$$

Since $4\lambda N^2\leq 4\lambda M^{1/3}<1$, we can apply Lemma 1 to the above terms and obtain

$$J(\lambda, M)\leq 2e^{2\lambda^2BN(2k+1)}\leq 2e^{2\lambda^2BM}.$$

If we put $2B=B_0$, we can prove the lemma.

For any integers $N\geq 0$ we consider the sequence $\{n_{N+k}\}, k=1,2,\cdots$, which satisfies (1.1), instead of $\{n_k\}, k=1,2,\cdots$, then for the same B_0 and M_0 as in Lemma 2 the following lemma holds.

Lemma 3. *Let $N\geq 0$ be any integer and positive λ and M satisfy*

(3.3) $$M>M_0 \text{ and } 4\lambda M^{1/3}<1,$$

then we have

$$\int_0^1 \exp\{\lambda\sum_{l=N+1}^{M+N}f(n_lt)\}dt\leq 2e^{\lambda^2B_0M}.$$

In the following let us put, for $M>0$ and $N\geq 0$,

(3.4) $$F(N, M, t)=\sum_{k=N+1}^{N+M}f(n_kt).$$

Lemma 4. *Let M and positive $\psi(M)$ satisfy the conditions*

(3.5) $$M>M_0 \text{ and } 16\psi(M)<B_0M^{1/3},$$

then we have

$$|\{t;\, 0\leq t\leq 1, F(N, M; t)>2\sqrt{B_0M\psi(M)}\}|\leq 2e^{-\psi(M)}.$$

Proof. If we put $\lambda=\sqrt{\psi(M)/B_0M}$, then (3.5) implies (3.3). Hence we have

$$|\{t;\, 0\leq t\leq 1, F(N, M; t)>2\sqrt{B_0M\psi(M)}\}|\leq 2e^{\lambda^2B_0M-2\lambda\sqrt{B_0M\psi(M)}}\leq 2e^{-\psi(M)}.$$

4. Proof of the theorem. To prove the theorem it is sufficient to show that

(4.1) $$\overline{\lim_{m\to\infty}}\frac{1}{\sqrt{2^mB_0\log m}}\sum_{k=1}^{2^m}f(n_kt)\leq 3, \quad \text{a.e. in } t,$$

and

(4.2) $$\overline{\lim_{m\to\infty}}\max_{2^m\leq n<2^{m+1}}\frac{1}{\sqrt{2^mB_0\log m}}\sum_{k=2^m+1}^{n}f(n_kt)\leq 3, \quad \text{a.e. in } t.$$

By Lemma 4 and the boundedness of $f(t)$ we can prove (4.1) and (4.2) in the same way as that of Erdös and Gál (c.f. §4 of the second paper of [I]).

References

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