59. On (m, n)-Mutant in Semigroup

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(Comm. by K. Kunugi, M.J.A., 12, 1962)

In his paper [2], A. A. Mullin introduced a new concept *mutant* concerned with biological computation (in particular, with physicological view) by Professor Heinz von Foester. For detail on biological computer and related subjects on relay switching field, see Engineering Outlook, the University of Illinois, vol. 1, no. 8 (1960) and A. A. Mullin [1].

Let S be a semigroup. A subset A in S is called *(m, n)-mutant* in S if $A^m \subset S - A^n$. The $(2, 1)$-mutant is a mutant in the sense of Mullin (2).

Proposition 1. *Every subset of a (m, n)-mutant of S is a (m, n)-mutant of S.*

Proof. Let B be a subset of the (m, n)-mutant A of S, then $B \subset A$ implies $B^k \subset A^k$ for every k. Hence

$$B^m \subset A^m \subset S - A^n \subset S - B^n.$$

Proposition 2. *Let $A_\alpha (\alpha \in \Lambda)$ be (m, n)-mutants of S, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a (m, n)-mutant of S, where $\bigcap_{\alpha \in \Lambda} A_\alpha$ is non-empty.*

Proof. From Proposition 1. Let φ be a homomorphism from S_1 into S_2, then $\varphi(a) \cdot \varphi(b) = \varphi(ab)$ for every $a, b \in S$.

Proposition 3. *Let A be a (m, n)-mutant of S_1. If $\varphi(S_1 - A^n) \subset S_2 - \varphi(A^n)$, then $\varphi(A)$ is a (m, n)-mutant in S_2.*

Proof. Proposition follows from

$$(\varphi(A))^m = \varphi(A^m) \subset \varphi(S_1 - A^n) \subset S_2 - \varphi(A^n) = S_2 - (\varphi(A))^n.$$

Proposition 4. *The inverse image under a homomorphism φ of a (m, n)-mutant is a (m, n)-mutant.*

Proof. Let homomorphism φ be φ: $S_1 \to S_2$, and suppose that B is a (m, n)-mutant of S_2. Let a $\in \varphi^{-1}(B)$, then we can find an element b in B such that $b = \varphi(a)$. Then $\varphi(a^m) = (\varphi(a))^m = b^m \in S_2 - B^n$. Hence $a^m \in \varphi^{-1}(S_2 - B^n) = S_1 - \varphi^{-1}(B^n) = S_1 - (\varphi^{-1}(B))^n$. This shows that $(\varphi^{-1}(B))^n \subset S_1 - (\varphi^{-1}(B))^n$.

A subset A of S is called to be a maximal (m, n)-mutant, if there is no (m, n)-mutant of S containing A.

By Zorn's lemma, we have

Proposition 5. *Every (m, n)-mutant is contained in a maximal (m, n)-mutant.*

We shall give an interesting example on (m, n)-mutant.

Let S be the set consisting of (a, b), where a, b are natural numbers. The product of elements (a, b) and (c, d) is defined as $(a, b)\cdot(c, d)=(ad+bc, bd)$. Then the multiplication is associative. Let $A=\{(a,1)\,|\,a\text{: natural number}\}$, then we have $A^2\subset S-A, A^4\subset S-A,\cdots,A^{2n}\subset S-A,\cdots, A^{2n-1}\subset A$ implies $A^{2m}\subset S-A^{2n-1}$. Therefore A is a $(2m, 2n-1)$-mutant. Similarly A is a $(2m-1, 2n)$-mutant. On the other hand A is neither a $(2m, 2n)$-mutant nor a $(2m-1, 2n-1)$-mutant.

References

[1] A. A. Mullin: The present theory of switching and some of its future trends, Industr. Math. Journal, **10**, 23-44 (1959-60).

[2] A. A. Mullin: A concept concerning a set with a binary composition law, Trans. of Illinois State Acad. of Sci., **53**, 144-145 (1960).