54. A Remark on Convexity Theorems for Fourier Series

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(Comm. by Z. SUETUNA, M.J.A., June 12, 1962)

In the previous paper [1], we have proved a number of convexity theorems concerning Fourier series. In the present paper, we shall improve some of them replacing either of the conditions by one-sided one.

Let $\varphi(t)$ be an even function integrable in $(0,\pi)$ in Lebesgue sense, periodic of period 2π, and let

$$\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt,$$

and

$$\Phi_0(t)=\varphi(t),\ \Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}\varphi(u)du\ (\alpha>0).$$

The (C,β) sum of the Fourier series of $\varphi(t)$ at $t=0$ is

$$s_n^\beta=A_n^\beta\frac{1}{2}a_0+\sum_{\nu=1}^{n}A_{n-\nu}^\beta a_\nu=\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-1}s_\nu\ (-\infty<\beta<\infty),$$

where $s_n=s_n^0$, $A_0^\beta=1$ and

$$A_n^\beta=\frac{(\beta+1)(\beta+2)\cdots(\beta+n)}{n!}\ (n\geqq 1).$$

In what follows we understand that $t\to 0$ means $t>0$ and $t\to 0$.

Now, Theorems 2, 4, 5, and 6 in the paper [1] can be improved as follows.

THEOREM 2′. *Let* $0\leqq b, 0<\beta-b\leqq\gamma-c$ *and* $|c-b|<1$. *If as* $t\to 0$,

(1) $$\int_0^t|\Phi_\beta(u)|du=o(t^{\gamma+1})$$

and

$$\int_0^t(|\Phi_b(u)|-\Phi_b(u))du=O(t^{c+1}),$$

then we have

$$s_n^r=o(n^q),\ q=b+(r-c)\frac{\beta-b}{\gamma-c},$$

as $n\to\infty$, *for* $c<r<\gamma'$, *where*

$$\gamma'=\inf\left(\gamma,\ \frac{(b+1)\gamma-(\beta+1)c}{\gamma-c+b-\beta}\right).$$

COROLLARY 2.1′. *Let* $0<\beta<\gamma$ *and* $0<\delta<1$. *If* (1) *holds, and* $\varphi(t)=O_L(t^{-\delta})$, *then*

$$s_n^\alpha=o(n^\alpha),\ \alpha=\beta\delta/(\gamma-\beta+\delta).$$

THEOREM 4′. *Let* $-1\leqq\beta, 0\leqq c$ *and* $0<\gamma+1-c\leqq\beta+1-b$,

$[b\gamma<(\beta+1)(c-1)]$. *If as* $n\to\infty$,

(2) $$\sum_{\nu=0}^{n}|s_\nu^\beta|=o(n^{\gamma+1})$$

and

$$\sum_{\nu=n}^{2n}(|s_\nu^{b-1}|-s_\nu^{b-1})=O(n^c),$$

then we have

$$\Phi_r(t)=o(t^q),\ q=b+(r-c)\frac{\beta+1-b}{\gamma+1-c},$$

as $t\to 0$, *for* $c<r<\gamma+1$.

COROLLARY 4.1′. *Let* $0<\delta<1$ *and* $-(1-\delta)<\gamma<\beta$. *If* (2) *holds, and* $a_n=O_L(n^{-(1-\delta)})$, *then*

$$\Phi_\alpha(t)=o(t^\alpha),\ \alpha=\delta(\beta+1)/(\beta-\gamma+\delta).$$

THEOREM 5′. *Let* $0\leqq b$ *and* $0<\beta-b\leqq\gamma-c$, $[(b-1)\gamma<c(\beta-1)]$. *If*

(3) $$\Phi_\beta(t)=o(t^\gamma) \text{ as } t\to 0,$$

and

$$\sum_{\nu=n}^{2n}(|s_\nu^{c-1}|-s_\nu^{c-1})=O(n^b) \text{ as } n\to\infty,$$

then we have

$$\Phi_r(t)=o(t^q),\ q=c+(r-b)\frac{\gamma-c}{\beta-b},$$

as t$\to 0$, *for* $b<r<\beta$.

COROLLARY 5′. *Let* $0<\delta<1$ *and* $\delta<\beta<\gamma$. If (3) *holds, and* $a_n=O_L(n^{-(1-\delta)})$, *then*

$$\Phi_\alpha(t)=o(t^\alpha),\ \alpha=\gamma\delta/(\gamma-\beta+\delta).$$

THEOREM 6′. *Let* $0\leqq c$, $0<\gamma-c\leqq\beta-b$ *and* $|b-c|<1$, $[c(\beta+1)<(b+1)\gamma]$. *If*

(4) $$s_n^\beta=o(n^\gamma) \text{ as } n\to\infty,$$

and

(5) $$\int_0^t(|\Phi_c(u)|-\Phi_c(u))du=O(t^{b+1}) \text{ as } t\to 0,$$

then we have

$$s_n^r=o(n^q),\ q=c+(r-b)\frac{\gamma-c}{\beta-b},$$

as $n\to\infty$, *for* $b<r<\beta$.

COROLLARY 6′. *Let* $0<\gamma<\beta$ *and* $0<\delta<1$. *If* (4) *holds, and* $\varphi(t)=O_L(t^{-\delta})$, *then*

$$s_n^\alpha=o(n^\alpha),\ \alpha=\gamma\delta/(\beta-\gamma+\delta).$$

Proof of Theorem 6′. Using the number γ' such as $\gamma'-c=\beta-b$ the assumptions imply

$$\gamma'>0 \text{ and } s_n^\beta=o(n^{\gamma'}).$$

So, by a theorem of Izumi [2], i.e. Corollary 4.2 in [1], we have $\Phi_{\gamma'+1+\varepsilon}(t)=o(t^{\beta+1+\varepsilon})$, $\varepsilon>0$. Consequently

(6) $$\Phi_{c+k}(t)=o(t^{b+k})$$
holds for every $k>\beta-b+1$. On the other hand, the condition (5) implies for $0<t\leqq t_0$

$$\int_t^{2t}(|\Phi_c(u)|-\Phi_c(u))du\leqq At^{b+1},$$

A being an absolute constant, and then

(7) $$\Phi_{c+1}(t+u)-\Phi_{c+1}(t)\geqq -At^{b+1},\ 0<u\leqq t.$$

From (6) and (7) we have $\Phi_{c+1}(t)=O(t^{b+1})$ by Theorem 8 in [1], and so (5) yields

(8) $$\int_0^t |\Phi_c(u)|\,du=O(t^{b+1}).$$

The result follows from (4) and (8). Cf. Theorem 6 and Lemmas 3 and 3′ in [1].

The proofs of Theorems 2′, 4′, and 5′ are similar.

References

[1] K. Yano: Convexity theorems for Fourier series, J. Math. Soc. Japan, **14**, 119–149 (1962), in the press.

[2] S. Izumi: Notes on Fourier analysis (XXVII): A theorem on Cesàro summation, Tôhoku Math. J. (2), **3**, 212–215 (1951).