149. On the Apriori Estimate for the Solution of Some Semi-Linear Wave Equation for Higher Space Dimension

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We have considered [1] already the equation of the following type;

$$\Delta u = u_{tt} + g(u) \tag{1}$$

and obtained an apriori estimate for the solution of the Cauchy problem for 3 space dimension under the condition:

$$\text{(2)}\qquad \begin{aligned} &\text{i)}\quad G(u)=\int_0^u g(u)du \geqq -L \\ &\text{ii)}\quad |g'(u)|\leqq c|u|^2 \quad (|u|\geqq k)\ c \text{ is a constant.} \end{aligned}$$

In this paper, we shall obtain the analogous results for the space dimension n higher than 3 assuming that the solution belongs to the space $D_{L^2}^{\left[\frac{n}{2}\right]+2}$. Our conditions for these cases are the following;

$$\text{(3)}\qquad \begin{aligned} &\text{i)}\quad G(u)=\int_0^u g(u) \geqq -L \quad (L>0) \\ &\text{ii)}\quad |g'(u)|\leqq c|u|^{\frac{2}{n-2}} \quad 2<n\leqq 6 \quad (|u|\geqq k) \\ &\text{iii)}\quad |g''(u)|\leqq M, \quad |g'''(u)|\leqq M_1. \end{aligned}$$

At first we introduce new unknown functions and we obtain a system of equations (4).

$$\text{(4)}\qquad \begin{aligned} &\frac{\partial u}{\partial t}=v \qquad \left(p_i=\frac{\partial u}{\partial x_i}\right) \\ &\frac{\partial v}{\partial t}=\sum_{i=1}^n p_{ix_i}-g(u) \\ &\frac{\partial p_i}{\partial t}=\frac{\partial v}{\partial x_i} \quad (i=1,2,\cdots,n). \end{aligned}$$

Our initial conditions for the Cauchy problem for u, v, p_i are $u(x,0)$, $v(x,0)$, $p_i(x,0)\in C_{0x}^{\left[\frac{n}{2}\right]+2}$, $C_{0x}^{\left[\frac{n}{2}\right]+1}$, $C_{0x}^{\left[\frac{n}{2}\right]+1}$ respectively and we suppose $g(u)\in C_u^3$; here we denote C_0^n the function space of the functions with continuous derivatives of n th order and with compact supports.

If we introduce an energy $E_0(t)$ of the solution u by the following integral form, we can easily prove its conservation, that is to say

$$E_0(t)=\int\left[G(u)+\frac{1}{2}v^2+\frac{1}{2}\sum_{i=1}^n p_i^2\right]dx \tag{5}$$

dx means n dimensional space element.

$$\frac{dE_0}{dt}=\int\Big[g(u)\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}+\sum_{i=1}^{n}p_i\frac{\partial p_i}{\partial t}\Big]dx$$
$$=+\int\Big[g(u)v+\sum_{i=1}^{n}p_{ix_i}v+\sum_{i=1}^{n}p_i\frac{\partial v}{\partial x_i}-g(u)v\Big]dx$$
$$=0$$

then, we have

(6) $\qquad E_0(t)=\text{const.}$ (the conservation of energy).

Next step is the estimation of energy of first order defined by

(7) $$E_1(t)=\frac{1}{2}\int\Big[\sum_{j=1}^{n}v_j^2+\sum_{i=1}^{n}p_{ij}^2\Big]dx,$$

where v_j means $\dfrac{\partial v}{\partial x_j}$ and p_{ij} means $\dfrac{\partial p_i}{\partial x_j}$ $\left(\text{that is } \dfrac{\partial^2 u}{\partial x_i\partial x_j}\right)$.

In order to obtain the estimation of this energy, differentiating the system (4) we have

(8) $$\begin{cases}\dfrac{\partial u_j}{\partial t}=v_j \\ \dfrac{\partial v_j}{\partial t}=\displaystyle\sum_{s=1}^{n}p_{ssj}-g'(u)p_j \qquad \left(p_{ssj}=\dfrac{\partial p_{ss}}{\partial x_j}\right) \\ \dfrac{\partial p_{ij}}{\partial t}=\dfrac{\partial v_j}{\partial x_i} \qquad \begin{pmatrix} i=1,\cdots,n \\ j=1,\cdots,n\end{pmatrix}\end{cases}$$

and we obtain,

$$\int_m|g'(u)p_jv_j|dx\leqq C\Big[\int|u|^{\frac{2n}{n-2}}\Big]^{\frac{1}{n}}\Big[\int|p_j|^{\frac{2n}{n-2}}\Big]^{\frac{n-2}{2n}}\Big[\int|v_j|^2\Big]^{\frac{1}{2}}$$

(m is the set of x for which $|u(x,t)|\geqq k$).

Applying the Sobolev's lemma for u and p_j, we obtain

$$\int_m|g'(u)p_jv_j|dx\leqq CE_0^{1/(n-2)}E_1^{1/2}E_1^{1/2}\leqq CE_0^{1/(n-2)}E_1$$

and obviously,

$$\int_{cm}|g'(u)p_jv_j|dx\leqq\frac{g_k}{2}E_0+\frac{g_k}{2}E_1 \qquad g_k=\operatorname*{Max}_{|u|\leqq k}|g'(u)|.$$

We have

$$\frac{dE_1}{dt}\leqq\Big(CE_0^{1/(n-1)}+\frac{g_k}{2}\Big)E_1+\frac{g_k}{2}E_0$$

where C, E_0, g_k are constants,

$$\frac{dE_1}{dt}\leqq C_1E_1+\frac{g_k}{2}E_0$$

(9) $$E_1(t)\leqq e^{c_1t}E_1(0)+\frac{E_0}{2}g_k\Big[\frac{e^{c_1t}}{C_1}-\frac{1}{C_1}\Big].$$

Third step is to obtain the estimation of the energy of 2nd order;

(10) $$E_2(t)=\frac{1}{2}\int\sum_{jk}[v_{jk}^2+\sum_{i=1}^{n}p_{ijk}^2]dx.$$

We differentiate the system (8) except the first equation.

$$(11)\qquad \begin{cases} \dfrac{\partial v_{kj}}{\partial t}=\sum_{s=1}^{n} p_{ssjk}-g'(u)u_{jk}-g''(u)u_k u_j \\ \dfrac{\partial p_{ijk}}{\partial t}=\dfrac{\partial v_{jk}}{\partial x_i} \qquad (i,j,k=1,2,\cdots,n) \end{cases}$$

and we have

$$\frac{dE_2}{dt}=-\sum_{j,k}^{n,n}\int g'(u)u_{jk}v_{kj}dx-\sum_{j,k}^{n,n}\int g''(u)u_k u_j v_{kj}dx.$$

i) At first, we estimate the first term ($u_{jk}=p_{jk}$).

$$\begin{aligned}&\int |g'(u)p_{jk}v_{jk}|dx\\ &\leqq\Big[c\int |u|^{\frac{2n}{n-2}}dx\Big]^{\frac{1}{n}}\Big[\int |p_{jk}|^{\frac{2n}{n-2}}dx\Big]^{\frac{n-2}{2n}}\Big[\int v_{kj}^2dx\Big]^{\frac{1}{2}}\\ &\leqq cE_0^{1/(n-2)}E_2(t)\end{aligned}$$

ii) $$\begin{aligned}\int |g''(u)u_k u_j v_{kj}|dx&=\int |g''(u)p_k p_j v_{kj}|dx\\ &\leqq M\Big[\int p_k^{\frac{n}{2}}dx\Big]^{\frac{2}{n}}\Big[\int p_j^{\frac{2n}{n-4}}dx\Big]^{\frac{n-4}{2n}}\Big[\int v_{jk}^2dx\Big]^{\frac{1}{2}}\\ &\leqq ME_1^{1/2}E_2(t) \qquad \text{for } \Big[\frac{n}{2}\leqq\frac{2n}{n-2} \text{ that is } n\leqq 6\Big]\end{aligned}$$

$$\frac{dE_2(t)}{dt}\leqq ME_1(t)^{1/2}E_2(t)+E_0^{1/(n-2)}E_2(t)$$

$$ME_1(t)^{1/2}+E_0^{1/(n-2)}=F(t) \quad \text{(bounded function)}$$

$$(12)\qquad E_2(t)\leqq E_2(0)e^{\int_0^t F(\tau)d\tau}.$$

Fourth step is the estimation of the energy of 3rd order;

$$(13)\qquad E_3(t)=\sum_{j,k,e}^{n,n,n}\frac{1}{2}\int\Big[v_{kje}^2+\sum_{i=1}^{n}p_{ijke}^2\Big]dx$$

Differentiating the system (11),

$$(14)\qquad \begin{aligned}&\frac{\partial v_{kje}}{\partial t}=\sum_{s=1}^{n}p_{ssjke}-g'(u)p_{jke}-g''(u)p_e p_{jk}\\ &\qquad -g''(u)p_{ke}p_j-g''(u)p_k p_{je}-g'''(u)p_e p_k p_j\\ &\frac{\partial p_{ijke}}{\partial t}=\frac{\partial v_{jke}}{\partial x_i} \qquad (i,j,k,e=1,2,\cdots,n)\\ &\frac{dE_3(t)}{dt}=\sum_{i,k,e=1}^{n}\Big[-\int g'(u)p_{jke}v_{kje}dx-l\int g''(u)p_{ke}p_j v_{jke}dx\\ &\qquad -\int g'''(u)p_e p_k p_j v_{jke}dx\Big].\end{aligned}$$

The question is to evaluate the third integrals,

$$\begin{aligned}&\int |g'''(u)p_e p_k p_j v_{jke}|dx\\ &\leqq M_1\int |p_e p_k p_j v_{jke}|dx\\ &\leqq M_1\Big[\int p_e^2p_k^2p_j^2dx\Big]^{\frac{1}{2}}\Big[\int v_{jke}^2dx\Big]^{\frac{1}{2}}\end{aligned}$$

$$\leqq M_1\left[\int p_e^{\frac{2n}{n-4}} dx\right]^{\frac{n-4}{2n}}\left[\int p_k^{\frac{2n}{n-4}} dx\right]^{\frac{n-4}{2n}}\left[\int p_j^{\frac{2n}{8-n}} dx\right]^{\frac{8-n}{2n}}\left[\int v_{jke}^2 dx\right]^{\frac{1}{2}}$$

$$\leqq M_1 E_2 E_3^{1/2} E_3^{1/2} \leqq M_1 E_2(t) E_3(t) \quad (4 < n \leqq 6).$$

Then we conclude

$$\frac{dE_3}{dt} \leqq F_1(t) E_3(t) \qquad (F_1(t) \text{ is one bounded function})$$

$$E_3(t) \leqq C_3 E_3(0). \tag{15}$$

Finally, we obtain the apriori bound for $u(x, t)$ by the Sobolev's lemma.

For $n \leqq 5$, $||u(x, t)||_2^{(3)}$ is bounded means that $u(x, t)$ is bounded.
$n = 6$, $||u(x, t)||_2^{(4)}$ is bounded means that $u(x, t)$ is bounded.

Bibliography

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