28. On the Propagation of Regularities of Solutions of Partial Differential Equations

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§1. **Introduction and Theorem.** Let $P\left(x, \frac{\partial}{\partial x}\right)$ be a partial differential operator of order m with C^∞-coefficients defined in a spherical neighbourhood $S(a, h)$ of a point a with radius h in R^n. Furthermore let Ω be a domain in R^n containing the point a as a point on its boundary.

Then we define the (a, Ω)-regularities as follows:

Definition 1. We say that $P\left(x, \frac{\partial}{\partial x}\right)$ is (a, Ω)-*regular* if for any r and any integer s, there are positive numbers l, t such that every distribution solution u of the equation $P(x, D)u=0$ defined in $S(a, r)$ is in $C^s(S(0, l))$, whenever $u \in C^t(S(a, r) \cap (R^n - \bar{\Omega}))$, where t depends upon r, l and s, but l depends only upon r and the derivatives of the coefficients of $P(x, D)$.

Definition 2. We say that $P\left(x, \frac{\partial}{\partial x}\right)$ is *hypo*-(a, Ω)-*regular* if for any r, there is a positive number l such that every distribution solution u of the equation $P(x, D)u=0$ defined in $S(a, r)$ is in $C^\infty(S(a, l))$, whenever $u \in C^\infty(S(a, r) \cap (R^n - \bar{\Omega}))$.

In the previous paper [4], I considered the case where $P(x, D)$ has constant coefficients and $\Omega = \{x \mid (x, \xi) > 0\}$ and showed that let $n \geqq 3$ then for a homogeneous polynomial p and for any q with order $\leq m-1$, $P=p+q$ is $(0, \Omega)$-regular if and only if the roots z of $p(z\xi+\eta)=0$ $(\xi \perp \eta)$ can't cut the real axis and the real roots are simple.

On the other hand in variable coefficients L. Hörmander considered every interesting theorems some of which showed the sufficient condition for P to be the hypo-Ω-regular [1].

In the present note under the same conditions as Hörmander's we shall generalize the distinctive feature of regularities mentioned above to variable coefficients as follows: denoting $p^{(j)}(x, \xi) = \frac{\partial}{\partial \xi_j} p(x, \xi)$, $\left(p_j(x, \xi) = \frac{\partial}{\partial x_j} p(x, \xi)\right.$ and $\bar{p}(x, \xi) = \overline{p(x, \xi)}$, we have

Theorem. Let the principal part $p(x, D)$ of $P(x, D)$ have the property such that for some real valued function $\phi \in C^\infty(S(0, h))$ with

$\left\{\dfrac{\partial \phi}{\partial x_i}(0)\right\} \neq \{0\}$,

$$(1)\qquad \operatorname{Re}\left\{\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial x_j \partial x_k} p^{(j)}(x,\xi)\overline{p^{(k)}(x,\xi)} + \sum_{j,k=1}^{n} [\overline{(p_j^{(k)}(x,\xi)}p^{(j)}(x,\xi) - p_j(x,\xi)\overline{p^{(jk)}(x,\xi)}]\frac{\partial \phi}{\partial x_k}\right\} < 0$$

if $x=0$ and $0\neq\xi\in R^n$ satisfy the characteristic equation

$$p(x,\xi)=0$$

and

$$\sum_{j=1}^{n} p^{(j)}(x,\xi)\frac{\partial \phi}{\partial x_j}=0.$$

Furthermore we assume that if for some $\xi\neq 0$, $p(0,\xi)=0$, then there exists a homogeneous polynomial $q(x,\xi)$ of order $m-1$ with $C^\infty(S(0,h))$-coefficients such that for $x\in S(0,h)$ and for $\xi\in R^n$

$$(2)\qquad i\{\overline{p^{(j)}(x,\xi)}p_j(x,\xi) - p^{(j)}(x,\xi)\overline{p_j(x,\xi)}\} = p(x,\xi)\overline{q(x,\xi)} + \overline{p(x,\xi)}q(x,\xi).$$

Then

(A): if the coefficients of p are real, $P(x,D)$ is $(0, \{x\,|\,\varphi(x)>0\}$-regular,

(B): if the coefficients of p are complex, $P(x,D)$ is not always $(0,\{x\,|\,\varphi(x)>0\})$-regular, but hypo-$(0,\{x\,|\,\varphi(x)>0\}$-regular,

(C): if the coefficients of P are analytic, then any distribution solution u of $P(x,D)u=0$ defined in $S(0,h)$ is in $C^\omega(S(0,l))$, whenever $u\in C^\omega(S(0,h)\frown\{x\,|\,\varphi(x)<0\})$, where l is independent of u.

Obviously conditions in the above theorem can be weakened ([4]), but it seems to be important that the conditions are invariant under the coordinate transformations and for the sake of simplicity in descriptions we omit here variations of Theorem.

§2. **Errata of my previous paper** [4]. P. 587. The condition in (2) of Corollary 2 read "the roots z of $p(z\xi+\eta)=0$ $(\xi\perp\eta)$ can't cut the real axis and the real roots are simple.

P. 587 and p. 588. "hypo-ξ-regular" in Corollaries 2 and 3 read "ξ-regular".

P. 588, line 31. "$x>-B$" read "$(x,\xi)>-B$".

§3. **The a priori estimate.** Using certain coordinate transformation we may assume that $\phi(x)=t$ $(x=(t,x^*)$, $x^*=(x_2,\cdots,x_n)$. Then we have the following Lemma which is weaker than Hörmander's [1], but it is direct to achieve our purpose.

Lemma. Under the hypothesis of Theorem, there exist a constant $K(>0)$ and a (sufficiently small) $l(>0)$ such that when α is sufficiently large

$$\alpha\sum_{|\beta|=m-1}\int |D^\beta u|^2 e^{-2\alpha g(t)}dx$$

$$(3)\qquad \leqq K\left\{\int |p(x,D)|^2 e^{-2\alpha g(t)}dx+q(\alpha)\sum_{|\beta|\leq m-2}\int |D^\beta u|^2 e^{-2\alpha g(t)}dx\right\}$$

$$u\in C_0^\infty(S(0,l)).$$

Where in the case A let $g(t)=\dfrac{L}{\delta-t}$ and in the case B let $g(t)=L(\delta-t)^\beta$ for some constant $L(>0)$, $\delta(\delta>l)$ and for $0<\beta<1$. Furthermore $q(\alpha)$ is a polynomial of α.

Proof. Setting $u=ve^{\alpha g(t)}$ we see that for $u\in C_0^\infty(S(0,\delta_0))$

$$e^{-\alpha g(t)}p(x,D)u=\sum_{j=0}^m \frac{1}{j!}\left(\frac{\partial^j}{\partial\xi_1^j}p(x,\xi)\right)_{\xi=\frac{\partial}{\partial x}} v g_j(t)$$

where

$$g_j(t)=\frac{d^j}{dt^j}(e^{\alpha g(t)})/e^{\alpha g(t)}.$$

Since $|g_j(t)|\leqq K\alpha^j$ for $0\leqq t\leqq\delta_0$ $(\delta>\delta_0)$, where from now on by K we denote constants independent of v and α, it implies that

$$\int |p(x,D)u|^2 e^{-2\alpha g(t)}dx\pm\int |\bar p(x,D)|^2 e^{-2\alpha g(t)}dx$$

$$=\int (|p(x,D)v|^2\pm|\bar p(x,D)v|^2)\,dx$$

$$+\int (|g_1(t)p^{(1)}v|^2\pm|g_1(t)\bar p^{(1)}v|^2)\,dx$$

$$+\int \{[(pv,g_1p^{(1)}v)+(g_1p^{(1)}v,pv)]\pm[(\bar pv,g_1\bar p^{(1)}v)+(g_1\bar p^{(1)}v,\bar pv)]\}\,dx$$

$$+\int \{[(pv,g_{2/2}p^{(1,1)}v)+(g_{2/2}p^{(1,1)}v,pv]\pm[\bar pv,g_{2/2}\bar p^{(1,1)}v)$$

$$+(g_{2/2}\bar p^{(1,1)}v,\bar pv)]\}\,dx+\varepsilon\alpha\sum_{|\beta|=m-1}\int |Dv|^2dx+K_\varepsilon\sum_{|\beta|\leq m-2}^{2(m-|\beta|)}\int |D^\beta v|^2dx,$$

where ε is an arbitrary small number and K_ε is a constant independent of v and α, but depends upon ε. Let $I_i^{(+)}$ and $I_i^{(-)}$ be the i-th term $(i=1,2,3,4)$ respectively of the right hand side of the above equalities. Then we see that the following inequalities are valid:

$$|I_1^{(-)}|\leqq\frac{1}{8\alpha}\left(\int |pv|^2dx+\int |\bar pv|^2dx\right)+\alpha C\sum_{|\beta|\leq m-1}\int |D^\beta v|^2dx$$

where C is a constant independent of α, v, and δ_0.
Furthermore

$$|I_2^{(-)}|\leqq\varepsilon\alpha\sum_{|\beta|=m-1}\int |D^\beta v|^2dx+\alpha^3K_\varepsilon\sum_{|\beta|\leq m-2}\int |D^\beta v|^2dx.$$

$$|I_3^{(-)}|\leqq 2\left[\left(\int |pv|^2dx\cdot\int |g_1p^{(1)}v|^2dx\right)^{\frac12}+\left(\int |\bar pv|^2dx\cdot\int |g_1\bar p^{(1)}v|^2dx\right)^{\frac12}\right.$$

$$I_3^{(+)}=\int ((\bar p^{(1)}(-D,x)g_1p(x,D)+\bar p(-D,x)g_1p^{(1)}(x,D)$$

$$+p^{(1)}(-D,x)g_1\bar p(x,D)+p(-D,x)g_1\bar p^{(1)}(x,D))v,v)\,dx$$

$$=\int ((-\bar p^{(1)}(-D,x)g_1'p^{(1)}(x,D)-p^{(1)}(-D,x)g_1'\bar p^{(1)}(x,D))v,v)\,dx+$$

$$+\int((-\overline{p}^{(1,j)}(-D,x)g_1p_j(x,D)-\overline{p}^{(1,j)}(-D,x)g_1\overline{p}_j(x,D)$$
$$-p^{(j)}(-D,x)g_1p_j^{(1)}(x,D)-p^{(j)}(-D,x)g_1\overline{p}_j^{(1)}(x,D))v,v)\,dx$$
$$+\int((-\overline{p}_j^{(1,j)}(-D,x)g_1p(x,D)-\overline{p}^{(1,1)}(-D,x)g_1'p(x,D)$$
$$-p_j^{(1,j)}(-D,x)g_1\overline{p}(x,D)-p^{(1,1)}(-D,x)g_1'\overline{p}(x,D))v,v)\,dx$$
$$+\int((-\overline{p}_j^{(j)}(-D,x)g_1p^{(1)}(x,D)-p_j^{(j)}(-D,x)g_1\overline{p}^{(1)}(x,D))v,v)\,dx$$
$$+\alpha K\sum_{|\beta|\leq m-2}\int|D^\beta v|^2\,dx$$
$$=I_{3,1}^{(+)}+I_{3,2}^{(+)}+I_{3,3}^{(+)}+I_{3,4}^{(+)}+I_{3,5}^{(+)}.$$

Where

$$I_{3,1}^{+}=\int(-g_1')(|p^{(1)}v|^2+|\overline{p}^{(1)}v|^2)\,dx,$$
$$|I_{3,3}^{+}|\leqq\frac{1}{8\alpha}\int(|pv|^2+|\overline{p}v|^2)\,dx+\sum_{|\beta|\leq m-2}\alpha^3K\int|D^\beta v|^2\,dx,$$
$$|I_4^{+}|,\ |I_4^{-}|\leqq\frac{1}{8\alpha}\int(|pv|^2+|\overline{p}v|^2)\,dx+\sum_{|\beta|\leq m-2}\alpha^5K\int|D^\beta v|^2\,dx.$$

Therefore we see that

$$2\int|P(x,D)u|^2e^{-2\alpha g(t)}\,dx$$
$$\geqq\Big[\Big(1-\frac{1}{\alpha}\Big)\int(|pv|^2+|\overline{p}v|^2)\,dx+\int g_1^2(|p^{(1)}v|^2+|\overline{p}^{(1)}v|^2)\,dx$$
$$-2\Big(\Big(\int|pv|^2\,dx\cdot\int|g_1p^{(1)}v|^2dx\Big)^{\frac{1}{2}}+\Big(\int|\overline{p}v|^2\,dx\cdot\int|g_1\overline{p}^{(1)}v|^2\,dx\Big)^{\frac{1}{2}}\Big)\Big]$$
$$+\frac{1}{2\alpha}\int(|pv|^2+|\overline{p}v|^2)\,dx$$
$$+I_{3,1}^{(+)}+I_{3,2}^{(+)}+I_{3,4}^{(+)}-\alpha(C+2\varepsilon)\sum_{|\beta|=m-1}\int|D^\beta v|^2\,dx$$
$$-K_\varepsilon\sum_{|\beta|\leq m-2}q(\alpha)\int|D^\beta v|\,dx.$$

The first term above inequality [ ]

$$\geqq-\frac{L_1}{\alpha}\int g_1^2(|p^{(1)}v|^2+|\overline{p}^{(1)}v|^2)\,dx.$$

Now we shall consider the bilinear form

$$\frac{1}{2\alpha}\int(|pv|^2+|\overline{p}v|^2)\,dx$$
$$+\int\Big(-g_1'-\frac{L_1}{\alpha}g_1^2\Big)\Big\{((\overline{p}^{(1)}(-D,x)p^{(1)}(x,D)+p^{(1)}(-D,x)\overline{p}^{(1)}(x,D))v,v)\,dx$$
$$+\int((-\overline{p}^{(1j)}(-D,x)g_1p_j(x,D)-p^{(1j)}(-D,x)g_1\overline{p}_j(x,D)$$
$$-\overline{p}^{(j)}(-D,x)g_1p_j{}^{(1)}(x,D)-p^{(j)}(-D,x)g_1\overline{p}_j^{(1)}(x,D))v,v)\,dx$$
$$+\int((-\overline{p}_j^{(j)}(-D,x)g_1p^{(1)}(x,D)-p_j^{(j)}(-D,x)g_1\overline{p}^{(1)}(x,D))v,v)\,dx.$$

Here we must remark that in the case B

$$-g_1'-\frac{L_1}{\alpha}g_1^2=L\alpha\beta\{(1-\beta)(\delta-t)^{\beta-2}-L_1\beta(\delta-t)^{2\beta-2}\geqq\frac{1}{\alpha}L_2g_1^2$$

for some sufficiently large constant L_2 if we take sufficiently small t and δ_0, for L_1 is independent of α, δ_0 and v. Furthermore in the case A, we can omit the term $I_i^{(-)}$, therefore we replace $\left(-g_1'-\frac{L_1}{\alpha}g^2\right)$ by $\left(\left(1-\frac{K}{\alpha}\right)g_1^2-g_1'\right)$, in the above bilinear form, which is also $\geqq\frac{L_2}{\alpha}g_1^2$.

Using the partition of unit with respect to the space $S(x,\delta_1)\times\{\xi\,|\,|\xi|=1\}$ ($\delta_1>\delta_0$) which is independent of α and applying Gårding's consideration and singular integral operators, we can prove from (1) that for a sufficiently large K_1 the above bilinear form

$$\geqq\alpha K_1\sum_{|\beta|=m-1}\int|D^\beta v|^2\,dx-\alpha^{2(m-1)}K_2\int|v|^2\,dx.$$

Thus returning from v to u we obtain the desired inequality. (Q.E.D.) Here we remark that to prove the above lemma we may replace $e^{-\alpha g(t)}$ by $e^{-\alpha t}$, but in the proof we must do $p(x,D)u$ by $g(t)\cdot p(x,D)u$ for certain $g(t)$. Thus Hörmander's estimate (1.1) [1] is not always valid in arbitrary bounded open sets under the conditions corresponding to (1) and (2) (see [4]).

§4. The proofs of (A) and (B) are accomplished from Lemma the consideration used in [1] and [2], but we must first consider in the case where $p\left(0,\frac{\partial\phi}{\partial x_i}\right)\neq 0$ and then use (1). Moreover in the proof of the counter part of B we use the result in [4]. The proof of (C) is also based upon Lemma. We have only to estimate the square integrations of each derivatives of solution u.

Finally we remark that Theorem is also valid in the large whenever conditions corresponding to the conditions (1) and (2) are satisfied on the domain which contain Ω.

References

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