27. On Conditionally Hypoelliptic Properties of Partially Hypoelliptic Operators

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1. Introduction. Recently L. Gårding and B. Malgrange [2, 3] have introduced the notions of partial hypoellipticity, partial ellipticity and conditional ellipticity. J. Friberg [1] and L. Hörmander [6] proved the fact that the solutions of $P(D)u=0$ is hypoanalytic of type σ in a fixed direction when $P(\zeta)$ is a polynomial of finite type σ in the same direction. J. Friberg also expected in his paper [1] that if $P(D)$ is partially hypoelliptic of type σ in some indepentent variables then the operator $P(D)$ have conditionally hypoelliptic properties in the same variables. (An operator $P(D)$ will be said to have a *conditionally hypoelliptic property of type σ in x'* if any solution $u\in A_{1(x'')}\cap C^{\infty}$ of $P(D)u=f(f\in A_{1(x)})$ belongs to $A_{\sigma(x)}$. See Def. 2.2.) The object of this note is to give a proof of above fact. The method is based on the idea of Gårding and Malgrange [2]. As the proof is somewhat mazy, details will be published later in the Osaka Mathematical Journal. I should like to thank Prof. M. Nagumo for his kind criticism during the preparation of this paper.

2. Algebraic considerations. Let $P(D)$ be a linear partial differential operator with constant coefficients operating on functions $u(x)$ defined in some open set $\Omega\subset R^m_{x'}\times R^n_{x''}$ $(x=(x',x'')=(x'_1,\cdots,x'_m,x''_1,\cdots,x''_n)\ x'\in R^m, x''\in R^n)$. By α we shall denote a multi-integer $(\alpha^{1'},\cdots,\alpha^{m'},\alpha^{1''},\cdots,\alpha^{n''})$ where $\alpha^{i'}$ and $\alpha^{j''}$ are non-negative integers, the length of α is denoted by $|\alpha|=\alpha^{1'}+\cdots+\alpha^{n''}$. Defining $D_{x'_j}=-\sqrt{-1}\,\partial/\partial x'_j$, $D_{x''_j}=-\sqrt{-1}\,\partial/\partial x''_j$ we set $D^{\alpha}=D^{\alpha'}_{x'}\cdot D^{\alpha''}_{x''}=D^{\alpha 1'}_{x'_1}\cdots D^{\alpha m'}_{x'_m}\cdot D^{\alpha 1''}_{x''_1}\cdots D^{\alpha n''}_{x''_n}$. By $P(\zeta)$ we mean the characteristic polynomial belonging to $P(D)$, and $V(P)$ denotes the algebraic variety in $C^m\times C^n$ defined by $\{\zeta;\ P(\zeta)=0\}\subset C^m\times C^n$.

Definition 2.1. The operator $P(D)$ (or $P(\zeta)$) is said to be *partially hypoelliptic of type σ in x'* if the following condition is satisfied.

There exist positive constants C_0 and σ (depending only on P) such that

$$(2.1)\qquad |Re\zeta'|\leqq C_0(1+|Im\zeta'|+|\zeta''|)^{\sigma}\quad (\zeta\in V(P))$$

or equivalently there exist positive constants C'_0 and σ for sufficiently large A

$$(2.1)'\qquad |Re\zeta'|\leqq C'_0(|Im\zeta'|+|\zeta''|)^{\sigma}\quad (\zeta\in V(P)\text{ and }|Re\zeta'|>A).$$

Remark 1. As in the proof of Lemma 3.9 in Hörmander [5], the best possible choice of above σ is always a rational number, therefore we may assume here $\sigma=r/s\,(\geqq 1)$ with mutually prime positive integer r and s.

Definition 2.2. A function $u(x)\in C^\infty(\Omega)$ is said to be *hypoanalytic of type σ in Ω* (we denote it $u(x)\in A_{\sigma(x)}(\Omega)$) if for every compact subset K of Ω there exists a positive constant C depending on K and u such that

(2.2) $$\operatorname*{Max.}_{x\in K}|D^p u(x)|\leqq C^{p+1}(p!)^\sigma \quad p=0,1,2,\cdots$$

is valid, where $|D^p u(x)|^2=\sum_{|\alpha|=p}\dfrac{p!}{\alpha'!\,\alpha''!}|D_{x'}^{\alpha'}D_{x''}^{\alpha''}u|^2$.

Lemma 2.1. $P(\zeta')(\zeta'\in C^m)$ is hypoelliptic of type σ: i.e.

(2.3) $$|Re\zeta'|\leqq C(1+|Im\zeta'|)^\sigma \quad (\zeta'\in V(p))$$

if and only if

(2.4) $$\sum_{|\alpha|>0}|P^{(\alpha)}(\xi')|^2|\xi'|^{2|\alpha|/\sigma}\leqq C'\sum_{|\alpha|\geqq 0}|P^{(\alpha)}(\xi')|^2 \quad (\xi'\in R^m)$$

or equivalently

(2.4)′ $$\sum_{|\alpha|>0}|P^{(\alpha)}(\xi')|^2|\xi'|^{2|\alpha|/\sigma}\leqq C''|P(\xi')|^2 \quad (|\xi'|>A').$$

Since $P(\zeta)=P(\zeta',\zeta'')$ is a polynomial in $C^m\times C^n$, P can be written as a finite sum;

(2.5) $$P(\zeta',\zeta'')=P_0(\zeta')+\sum_{|\gamma|>0}P_\gamma(\zeta')\cdot(\zeta'')^\gamma$$

where $\gamma=(\gamma^1,\cdots,\gamma^n)$ with non negative integer γ^i. Then the following theorem is established.

Theorem 2.1. *$P(\zeta)$ is partially hypoelliptic of type σ in x' if and only if*

(2.6) $$\sum_{|\alpha+\gamma|\geqq 0}|P_\gamma^{(\alpha)}(\xi')|^2|\xi'|^{2|\alpha+\gamma|/\sigma}\leqq C_1(|P_0(\xi')|^2+1) \quad (\xi'\in R^m).$$

Remark 2. If $P(\zeta)$ is partially hypoelliptic of type σ in x' then by virtue of (2.1) $P_0(\zeta')(=P(\zeta',0))$ is *hypoelliptic of type σ* as a polynomial in ζ'. Hence the following inequality is valid.

(2.7) $$\sum_{|\alpha|>0}|P_0^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha|/\sigma}\leqq C_2|P_0(\xi')| \quad (\xi'\in R^m,\ |\xi'|>A').$$

It is easily verified that (2.6) is equivalent to

(2.6)′ $$\sum_{\substack{|\alpha+\gamma|>0\\|\alpha|\geqq 0}}|P_\gamma^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha+\gamma|/\sigma}\leqq C_1'|P_0(\xi')| \quad (|\xi'|>A'')$$

or

(2.6)″ $$\sum_{\substack{|\alpha+\gamma|>0\\|\alpha|\geqq 0}}|P_\gamma^{(\alpha)}(\xi')|^{2r}|\xi'|^{2s|\alpha+\gamma|}\leqq C_1''(|P_0(\xi')|^{2r}+1) \quad (\xi'\in R^m).$$

Proof of Theorem 2.1. Writing $\zeta'=\xi'+i\eta'$ $(\xi',\eta'\in R^m\ i=\sqrt{-1})$ (2.5) can be written as follows:

(2.8) $$P(\zeta)=P_0(\xi')+\sum_{|\alpha|>0}C_\alpha P_0^{(\alpha)}(\xi')(i\eta')^\alpha+\sum_{|\gamma|>0}\sum_{|\alpha|>0}C_\alpha P_\gamma^{(\alpha)}(\xi')(i\eta')^\alpha(\zeta'')^\gamma$$
$$(C=\max_{0\leqq|\alpha|\leqq\rho}C_\alpha,\ \rho=\text{degree of } P).$$

Let $\eta'=|\xi'|^{1/\sigma}\tilde{\eta}'$, $\zeta''=|\xi'|^{1/\sigma}t\cdot\tilde{\zeta}''$ where $\tilde{\eta}'\in R^m$, $\tilde{\zeta}''\in C^n(|\tilde{\zeta}''|=1)$, $t\in C^1$ and $t\cdot\tilde{\zeta}''=(t\cdot\tilde{\zeta}_1'',\cdots,t\cdot\tilde{\zeta}_n'')$, then (2.8) is transformed into

$$(2.9)\qquad P(\zeta)=P_0(\xi')+\sum_{|\alpha|>0}C_\alpha P_0^{(\alpha)}(\xi')|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha+\sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0}C_\alpha P_\gamma^{(\alpha)}(\xi')|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha(\tilde{\zeta}'')^\gamma t^{|\gamma|}.$$

Now first of all fix the length of $\tilde{\eta}'(=\varepsilon)$ suitably (for example; $|\tilde{\eta}'|=\frac{1}{2}\mathrm{Min}\{(C_0)^{-1},(\overline{C}C_2)^{-1},1\}$) then according to (2.7) there exist constants C_3, C_3' such that

$$(2.10)\qquad C_3|P_0(\xi')|\leqq|P_0(\xi')+\sum_{|\alpha|>0}C_\alpha P_0^{(\alpha)}(\xi')|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha|\leqq C_3'|P_0(\xi')|\qquad(|\xi'|>A').$$

Thus according to the condition (2.1)′, if $t\in C^1$ is a solution of

$$(2.11)\qquad P_0(\xi')+\sum_{|\alpha|>0}C_\alpha P_0^{(\alpha)}(\xi')|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha+\sum_{|\gamma|>0}\sum_{|\alpha|\geqq 0}C_\alpha P_\gamma^{(\alpha)}(\xi')|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha(\tilde{\zeta}'')^\gamma t^{|\gamma|}=0$$

then $|t|>C_4$ for some positive C_4 uniformly in $\tilde{\eta}'\in R^m(|\tilde{\eta}'|=\varepsilon)$, $\tilde{\zeta}''\in C^n\ (|\tilde{\zeta}''|=1)$ and $|\xi'|>A'$. This shows that every solution τ of

$$(2.11)'\qquad \tau^\rho+\sum_{k=1}^{\rho}\sum_{|\gamma|=k}\left\{\frac{\sum_{|\alpha|\geqq 0}C_\alpha P_\gamma^{(\alpha)}(\xi')|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha}{\sum_{|\alpha|\geqq 0}C_\alpha P_0^{(\alpha)}(\xi')|\xi'|^{|\alpha|/\sigma}(i\tilde{\eta}')^\alpha}\right\}(\tilde{\zeta}'')^\gamma\tau^{\rho-k}=0$$

satisfies $|\tau|<1/C_4$ uniformly.

This shows that every coefficient of $\tau^k(k=0,\cdots,\rho-1)$ is uniformly bounded. By virtue of uniformity in $\tilde{\zeta}''$, and (2.10)

$$\{\sum_{|\alpha|\geqq 0}P_\gamma^{(\alpha)}(\xi')|\xi'|^{|\alpha+\gamma|/\sigma}(i\tilde{\eta}')^\alpha\}/|P_0(\xi')|$$

is uniformly bounded in $\tilde{\eta}'(|\tilde{\eta}'|=\varepsilon)$ and $\xi'\ (|\xi'|>A)$.

Finally from the uniformity in $\tilde{\eta}'(|\tilde{\eta}'|=\varepsilon)$ the result follows.

It is easily verified by the well-known method that (2.6) implies (2.1) (cf. p. 28, [1]).

3. A priori estimates. In this section we introduce a new norm (similar as introduced in [1]) which depend on the operator $P(D)$ and δ with $0<\delta\leqq 1$.

Let K be any given relatively compact subset in $\Omega\subset R^m\times R^n$ with $\overline{K}\subset\Omega$. We then define the norm of $u\in C^\infty(\Omega)$ as follows:

$$(3.1)\qquad |u,K|_\delta^2=\sum_{|\gamma|\geqq 0}\sum_{\alpha_i,k}\|Q_\gamma^{(\alpha_1)}(D)\cdots Q_\gamma^{(\alpha_r)}(D)\cdot D^k u,K\|^2\delta^{2\sigma k-2\Sigma|\alpha_i|}$$

where $Q_\gamma(D)=P_\gamma(D_{x'})D_{x''}^\gamma$ and $\|f,K\|$ denotes the usual L^2 norm of f on K.

The sum is to be taken over all index sets $\alpha_i=(\alpha_i',\alpha_i'')$ with $0<|\alpha_1|\leqq|\alpha_2|\leqq\cdots\leqq|\alpha_r|\leqq\rho$ ($\rho=\deg P$) and over all integers k with

$$0=k<s\cdot\min|\alpha_i|=s|\alpha_1|.$$

By the definition, the exponent of δ is always negative and the highest order derivatives of u contained in $|u,K|_\delta^2$ is smaller than $r\cdot\rho-(r-s)$. Therefore the following inequalities are valid.

$$(3.2)\qquad C_5\sum_{0\leqq k<s\cdot\rho}\|D^k u,K\|^2\leqq|u,K|_1^2\leqq C_6\sum_{|\alpha|\leqq r\cdot\rho-(r-s)}\|D^\alpha u,K\|^2$$

for some C_5, C_6 which do not depend on u and δ.

(3.3) $$|u, K|_1 \leqq |u, K|_\delta = |u, K|_1 \cdot \delta^{-r\cdot\rho}.$$

Lemma 3.1. Let K_0, K_1 be relatively compact subdomains in Ω with

$$K_0 \subset K_1 \subset \overline{K}_1 \subset \Omega \text{ and dist.}(\partial K_0, \partial K_1) = \delta > 0.$$

Then there exists a $\varphi(x) \in C_0^\infty(K_1)$ with properties; $\varphi(x) \geqq 0$ on K_1, $\varphi(x) = 1$ on K_0 and

(3.4) $$|D^\alpha \varphi(x)| \leqq \widetilde{C} \delta^{-|\alpha|} (x \in K_1, |\alpha| \leqq r\cdot\rho).$$

Lemma 3.2. If $R_i(\xi)$ is a polynomial with constant coefficients then

(3.5) $$\|R_1(D)\cdots R_r(D)v(x)\|^2 = r^{-1}\sum_{i=1}^{r}\|R_i(D)^r v(x)\|^2 \quad (v \in C_0^\infty).$$

Theorem 3.1. *Let $P(D)$ be a partially hypoelliptic operator of type σ in x' and K_0, K_1 be relatively compact subdomains of Ω with $K_0 \subset K_1 \subset \overline{K}_1 \subset \Omega$ such that* dist. $(\partial K_0, \partial K_1) = \delta$ $(0 < \delta \leqq 1)$.

Then there exists a constant C_7 (independent of u and δ), such that

(3.6) $$\delta^\sigma |Du, K_0|_\delta \leqq C_7 \left\{ \sum_{k=0}^{r-s+1} |D_{x''}^k u, K_1|_\delta \delta^k + \sum_{0 \leqq |\alpha| \leqq \rho(r-1)} \|D^\alpha P(D)u, K_1\| \delta^{-\rho(r-1)} \right\}$$

for all $u \in C^\infty(\Omega)$.

(*Outline of Proof.*) The quantity that we are going to estimate is

(3.7) $$\delta^{2\sigma}|Du, K_0|_\delta^2 = \sum_{|r| \geqq 0} \sum_{\substack{0 < |\alpha_1| \leqq \cdots \leqq |\alpha_r| \\ 0 \leqq k < s|\alpha_1|}} \|Q_r^{(\alpha_1)}(D)\cdots Q_r^{(\alpha_r)}(D)D^{k+1}u, K_0\|^2 \delta^{2\sigma(k+1) - 2\Sigma|\alpha_i|}.$$

We can split the above sum into two parts so that in the first part $k+1 < s|\alpha_1|$, while in the second $k+1 = s|\alpha_1|$, then

(3.8) $$\text{The 1st part } \leqq C_8 |u, K_0|_\delta^2 \leqq C_8 |u, K_1|_\delta^2.$$

In the second each term is estimated as follows (if we set $v = \varphi \cdot u \in C_0^\infty(K_1)$ and using Lemma 3.2).

(3.9) $$\|Q_r^{(\alpha_1)}(D)\cdots Q_r^{(\alpha_r)}(D)\cdot D^{s|\alpha_1|}u, K_0\|^2 \delta^{2\sigma s|\alpha_1| - 2\Sigma|\alpha_i|} \leqq r^{-1}\sum_{i=1}^{r}\|Q_r^{(\alpha_i)}(D)^r D^{s|\alpha_1|}v, K_1\|^2 \delta^{-2r(|\alpha_i| - |\alpha_1|)}.$$

The right hand side of (3.9) is composed of the terms of two different types,

(3.10) $$\|Q_r^{(\alpha)}(D)^r D^{s|\alpha|}v\|^2$$

(3.11) $$\|Q_r^{(\alpha)}(D)^r D^{sk}v\|^2 \delta^{-2r(|\alpha| - k)} \quad (|\alpha| > k).$$

Then after some calculations we have

(3.11)′ $$\|Q_r^{(\alpha)}(D)^r D^{sk}v\|^2 \delta^{-2r(|\alpha| - k)} \leqq C_9 |u, K_1|_\delta^2$$

and

(3.10)′ $$\|Q_r^{(\alpha)}(D)^r D^{s|\alpha|}v\|^2 = \sum_{k=0}^{s|\alpha|}\binom{s|\alpha|}{k}\int |Q_r^{(\alpha)}(\xi)|^{2r}|\xi'|^{2(s|\alpha|-k)}|\xi''|^{2k}|v(\xi)|^2 d\xi.$$

Every term in (3.10)′ with $k \geqq 1$ is estimated by

(3.12) $$C_{10}\{|u, K_1|_\delta^2 + |D_{x''}u, K_1|_\delta^2\delta^2\}.$$

Finally we shall estimate the quantity

(3.13) $$\int |Q_\gamma^{(\alpha)}(\xi)|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2 d\xi \leqq C\int |P_\gamma^{(\alpha')}(\xi')(\xi'')^{\gamma-\alpha''}|^{2r}|\xi'|^{2s|\alpha|}|\hat{v}(\xi)|^2 d\xi$$

in two cases. The first case. $\alpha=(\alpha',\alpha'')$, $|\gamma-\alpha''|>0$

(3.14) $$(3.13)\leqq C_{11}\sum_{0\leqq k\leqq r-s+1}|D_{x''}^k u, K_1|_\delta^2\delta^{2k}.$$

The second case: $\alpha''=\gamma$. Using (2.6)″ we have

(3.15) $$(3.13)\leqq C_1\Big\{\int|\hat{v}(\xi)|^2 d\xi + \int |P(\xi)-\sum_{|\gamma|>0}P_\gamma(\xi')(\xi'')^\gamma|^{2r}|\hat{v}(\xi)|^2 d\xi \leqq C_{12}\Big\{\sum_{0\leqq k\leqq r-s+1}|D_{x''}^k u, K_1|_\delta^2\delta^{2k} + \sum_{|\alpha|\leqq\rho(r-1)}\|D^\alpha P(D)u, K_1\|^2\delta^{-2\rho(r-1)}\Big\}.$$

(In this proof constants C_s' are independent of u and δ.) Therefore (3.8) (3.10) (3.12) (3.14) and (3.15) show the theorem.

Corollary 3.1. *Let $P(D)$ be a partially hypoelliptic operator of type σ in x', ρ be the degree of $P(\zeta)$, and K and L be arbitrary relatively compact subdomains of Ω such that $K\subset L\subset\bar{L}\subset\Omega$ and* dist. $(\partial K,\partial L)=\delta$ $(0<\delta\leqq 1)$. *Then there exists a constant C_{13} such that the inequality*

(3.16) $$(\delta/p)^{p\sigma}|D^p u, K|_{\delta/p}\leqq C_{13}^p\Big\{\sum_{k=0}^{r-s+1\cdot p}(\delta/p)^k|D_{x''}^k u, L|_{\delta/p} + \sum_{k=0}^{p(r-s+1)}\sum_{|\alpha|\leqq\rho(r-1)}\|D^\alpha P(D)\cdot D^k u, L\|(\delta/p)^{k-\rho(r-1)}\Big\}\ p=0,1,2,\cdots$$

is valid for all $u\in C^\infty(\Omega)$.

The constant C_{13} does not depend on p.

Proof. By the assumptions on K and L there exists an increasing sequence of relatively compact domains $K_0, K_1,\cdots,K_p$ such that

$$K=K_0\subset K_1\cdots\subset K_p=L \text{ and dist.}(\partial K_i,\partial K_{i+1})=\delta/p<1.$$

Thus every pair K_i, K_{i+1} satisfies the conditions imposed on K_0 and K_1 in Theorem 3.1. If $u\in C^\infty(\Omega)$ then for every $i=0,1,\cdots,D^i u\in C^\infty(\Omega)$. Successive applications of Theorem 3.1 to K_i, K_{i+1} show the inequality (3.16).

Now Corollary 3.1 and Sobolev's lemma lead to the following

Main Theorem. *Let $P(D)$ be a partially hypoelliptic operator of type σ in x' and $u(\in C^\infty(\Omega))$ be a solution of $P(D)u=f$ $(f\in A_{1(x)})$ in Ω such that $D^k u\in A_{1(x'')}$ for every $k(k=0,1,\cdots,\rho r-(r-s)-1)$. Then u belongs to $A_{\sigma(x)}$.*

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