26. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. VI*

By Sakuji INOUE
Faculty of Education, Kumamoto University

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On the assumption that ζ and Ω denote respectively a given complex number and an appropriately large circle with center at the origin and that the ordinary part $R(\lambda)$ of the function $S(\lambda)$ defined in the statement of Theorem 1 [1] is a transcendental integral function, in this paper we shall discuss the relation between the distribution of ζ-points of $S(\lambda)$ and that of ζ-points of $R(\lambda)$ in the exterior of the same circle Ω and shall then show that, if each of $S(\lambda)$ and $R(\lambda)$ has its finite exceptional value for the exterior of Ω, the two exceptional values are identical under some conditions.

Theorem 16. Let $S(\lambda)$, $R(\lambda)$, and $\{\lambda_\nu\}$ be the same notations as those in Theorem 1; let σ be an appropriately large number such that $\sup_\nu|\lambda_\nu|<\sigma<\infty$; let $\{z_n\}$ be an infinite sequence of all ζ-points of $R(\lambda)$ in the exterior of the circle $|\lambda|=\sigma$ such that

$$\left.\begin{aligned}&R(z_n)=\zeta\\&\sigma<|z_n|\leqq|z_{n+1}|\end{aligned}\right\}\ (n=1,2,3,\cdots)$$

and $|z_n|\to\infty$ $(n\to\infty)$, each ζ-point being counted with the proper multiplicity; let

$$C=\sup_n\left\{\frac{1}{2\pi}\left|\int_0^{2\pi}S(\rho e^{it})e^{int}dt\right|\right\}\ (<\infty),$$

where ρ is an arbitrarily prescribed number subject to the condition $\sup_\nu|\lambda_\nu|<\rho<\infty$; let μ be the greatest value of the positive integers ν_n in the first non-zero coefficients $R^{(\nu_n)}(z_n)/\nu_n!$ of the Taylor expansions of $R(\lambda)$ at z_n, $n=1,2,3,\cdots$; let $m\equiv\inf_n\{|R^{(\nu_n)}(z_n)|/\nu_n!\}$ be positive; let $M\equiv\sup_n[\max_k\{|R^{(k)}(z_n)|/k!\}]$ $(n,k=1,2,3,\cdots)$ be finite; and let r be an arbitrarily given number such that $0<r<m/(M+m)$. Then, in the interior of the circle $|\lambda-z_n|=r$ associated with any z_n satisfying

$$\left\{\frac{C}{r^\mu\left(m-\dfrac{Mr}{1-r}\right)}+1\right\}\rho+r<|z_n|,$$

$S(\lambda)$ has ζ-points whose number (counted according to multiplicity) equals that of ζ-points of $R(\lambda)$ in the interior of the same circle as it.

Proof. It must first be noted that the case where $R(\lambda)$ has such ζ-points $\{z_n\}$ as was described in the statment of the present theorem can occur in accordance with Picard's theorem when it is a transcendental integral function.

Now, by hypotheses,

$$|R(z_n+re^{i\theta})-\zeta|=\left|\sum_{k=1}^{\infty}\frac{R^{(k)}(z_n)}{k!}(re^{i\theta})^k\right|$$
$$\geqq r^{\nu_n}\left(m-\frac{Mr}{1-r}\right)$$
$$\geqq r^{\mu}\left(m-\frac{Mr}{1-r}\right)>0,$$

where ν_n is the same notation as that defined in the statement of the present theorem; and in addition, denoting by $\chi(\lambda)$ the sum of the two principal parts of $S(\lambda)$ and applying the expansions of $R(\lambda)$ and $S(\lambda)$ [2], we can find at once that for every z_n satisfying $|z_n| > r+\rho$

$$\left|\chi(z_n+re^{i\theta})\right|=\frac{1}{2}\left|\sum_{k=1}^{\infty}(a_k+ib_k)\left(\frac{\rho}{z_n+re^{i\theta}}\right)^k\right|$$
$$\leqq\frac{1}{2}\sum_{k=1}^{\infty}|a_k+ib_k|\left(\frac{\rho}{|z_n|-r}\right)^k$$
$$\leqq\frac{C\rho}{|z_n|-r-\rho}<\infty,$$

where

$$\left.\begin{aligned}a_k&=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})\cos kt\,dt\\ b_k&=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})\sin kt\,dt\end{aligned}\right\}.$$

Since, on the other hand, there exist large positive integers n such that

$$0<\frac{C_\rho}{|z_n|-r-\rho}<r^{\mu}\left(m-\frac{Mr}{1-r}\right),\ \text{i.e.,}\left\{\frac{C}{r^{\mu}\left(m-\dfrac{Mr}{1-r}\right)}+1\right\}\rho+r<|z_n|,$$

by denoting by G the least value of n satisfying this last inequality we obtain the inequalities $|R(z_{G+p}+re^{i\theta})-\zeta|>|\chi(z_{G+p}+re^{i\theta})|$, $p=0,1,2,\cdots$, for every θ in the closed interval $[0, 2\pi]$. If, for simplicity, we denote by Γ_p the circle $|\lambda-z_{G+p}|=r$ associated with the point z_{G+p} for each value of $p=0,1,2,\cdots$, then the just established result shows that $|R(\lambda)-\zeta|>|\chi(\lambda)|$ on Γ_p, $p=0,1,2,\cdots$. In addition to it, $R(\lambda)-\zeta$ and $\chi(\lambda)$ are both regular inside and on any Γ_p by the condition $|z_{G+p}|>r+\rho$. In consequence, it is found with the help of Rouché's theorem that the function $S(\lambda)-\zeta=\{R(\lambda)-\zeta\}+\chi(\lambda)$ has zeros (with multiplicities properly counted) inside any Γ_p and that the number of those zeros is equal to that of zeros (with multiplicities properly counted) of $R(\lambda)-\zeta$ inside the same Γ_p. Evidently this implies that the result stated in the present theorem holds true.

Theorem 17. Let $S(\lambda)$, $R(\lambda)$, $\{\lambda_\nu\}$, σ, ρ, C, μ, m, M, and r be the same notations as those in Theorem 16 but let $\{z_n\}$ in it be an infinite sequence of all ζ-points of $S(\lambda)$ in the exterior of the circle $|\lambda|=\sigma$ such that

$$\left.\begin{array}{l} S(\lambda)=\zeta \\ \sigma<|z_n|\leqq|z_{n+1}| \end{array}\right\} \quad (n=1, 2, 3, \cdots)$$

and $|z_n|\to\infty$ $(n\to\infty)$, each ζ-point being counted with the proper multiplicity; and let ε be a positive number less than $r^\mu\left(m-\dfrac{Mr}{1-r}\right)$. Then, in the interior of the circle $|\lambda-z_n|=r$ associated with any z_n satisfying the conditions $|R(z_n)-\zeta|<\varepsilon$ and

$$\left\{\frac{2C}{r^\mu\left(m-\dfrac{Mr}{1-r}\right)-\varepsilon}+1\right\}\rho+r<|z_n|,$$

$R(\lambda)$ has ζ-points whose number (counted according to multiplicity) equals that of ζ-points of $S(\lambda)$ in the interior of the same circle as it.

Proof. As will be seen immediately from the expansion of $\chi(\lambda)$ [2], $|\chi(\lambda)|\to 0$ $(|\lambda|\to\infty)$ and so $|R(z_n)-\zeta|\to 0$ $(n\to\infty)$ by virtue of the hypothesis $S(z_n)=\zeta$, $n=1, 2, 3\cdots$. Since, moreover, by hypotheses,

$$\begin{aligned} |R(z_n+re^{i\theta})-\zeta| &\geqq r^\mu\left(m-\frac{Mr}{1-r}\right)-|R(z_n)-\zeta| \\ &> r^\mu\left(m-\frac{Mr}{1-r}\right)-\varepsilon \end{aligned}$$

for all z_n with $|R(z_n)-\zeta|<\varepsilon$, and since, as demonstrated in the course of the proof of Theorem 16,

$$|\chi(z_n+re^{i\theta})|\leqq\frac{C\rho}{|z_n|-r-\rho}<\infty$$

for any z_n with $|z_n|>r+\rho$, it can be verified without difficulty from the relation $S(z_n+re^{i\theta})-\zeta=\{R(z_n+re^{i\theta})-\zeta\}+\chi(z_n+re^{i\theta})$ that $|S(z_n+re^{i\theta})-\zeta|>|\chi(z_n+re^{i\theta})|$ for every $\theta\in[0, 2\pi]$ and every z_n satisfying the conditions $|R(z_n)-\zeta|<\varepsilon$ and

$$0<\frac{2C\rho}{|z_n|-r-\rho}<r^\mu\left(m-\frac{Mr}{1-r}\right)-\varepsilon, \text{ i.e.,}$$

$$\left\{\frac{2C}{r^\mu\left(m-\dfrac{Mr}{1-r}\right)-\varepsilon}+1\right\}\rho+r<|z_n|.$$

For any z_n satisfying these two conditions, we have therefore the inequality $|S(\lambda)-\zeta|>|\chi(\lambda)|$ holding on the circle $|\lambda-z_n|=r$, and moreover $S(\lambda)-\zeta$ and $\chi(\lambda)$ are both regular inside and on this circle by the condition $|z_n|>r+\rho$. On the other hand, as can be seen from the familiar method of the proof of the Rouché theorem quoted before, it is rewritten as follows: if $f(\lambda)$ and $g(\lambda)$ are both regular on a simply connected domain D, if Γ is the curve defined by the equation $\lambda=\xi(s)$, $(0\leqq s\leqq 1, \xi(0)=0, \xi(1)=1)$, where $\xi(s)$ is a continuous function of s, and if for any point ξ on Γ the function $f(\lambda)-\xi(s)g(\lambda)$ never vanishes on a rectifiable closed Jordan curve K contained in D, then, in the interior of K, the number (counted according to multiplicity) of zeros of $f(\lambda)-g(\lambda)$ coincides with that of zeros of $f(\lambda)$. In consequence, by applying this rewritten Rouché theorem to the above established results, we can conclude that the number (counted according to multiplicity) of ζ-points of the function $R(\lambda)=S(\lambda)-\chi(\lambda)$ inside any circle $|\lambda-z_n|=r$ where z_n satisfies the above-mentioned conditions is equal to that of ζ-points of $S(\lambda)$ inside the same circle as it.

The present theorem has thus been proved.

Theorem 18. Let $S(\lambda)$, $R(\lambda)$, $\{\lambda_\nu\}$, and σ have the same meanings as in Theorems 16 and 17 respectively. If $S(\lambda)$ has $\zeta(\neq\infty)$ as its exceptional value for the exterior of the circle $|\lambda|=\sigma$, that is, if the equation $S(\lambda)=\zeta$ has not infinitely many solutions in the domain $\mathfrak{D}\{\lambda: |\lambda|>\sigma\}$, then the same is also valid of the equation $R(\lambda)=\zeta$, and conversely.

Proof. First we consider the case where $S(\lambda)$ has ζ as its finite exceptional value for the above-mentioned domain $\mathfrak{D}$. If, contrary to what we wish to prove, ζ is not the exceptional value of $R(\lambda)$ for $\mathfrak{D}$, there would exist ζ-points $\{z_n\}_1^\infty$ of $R(\lambda)$, which are so arranged as to satisfy the conditions stated in Theorem 16. Contrary to the hypothesis on $S(\lambda)$, this result would lead us to the conclusion that $S(\lambda)$ has also an infinite sequence of ζ-points in $\mathfrak{D}$, according to Theorem 16. Consequently ζ must be the exceptional value of $R(\lambda)$.

Next we consider the case where ζ is the finite exceptional value of $R(\lambda)$. In this case, by making use of a method analogous to that applied in the preceding paragraph and of Theorem 17 it can be verified similarly that $S(\lambda)$ has ζ as its exceptional value for the domain $\mathfrak{D}$.

The proof of the theorem is thus complete.

Remark. We here remark on $R^{(k)}(\lambda)$, $k=0,1,2,\cdots$, that each of these functions is expressible by a curvilinear integral associated with $S(\lambda)$ itself, as shown in Theorem 1.

References

[1] S. Inoue: Some applications of the functional-representations of normal operators in Hilbert spaces, Proc. Japan Acad., **38**, 265–266 (1962).
[2] ——: Some applications of the functional-representations of normal operators in Hilbert spaces. III, Proc. Japan Acad., **38**, 641–642 (1962).

Correction to Sakuji Inoue: "Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. V" (Proc. Japan Acad., **38**, 706–710 (1962)).

Page 707, line 6 from bottom:

For "$\frac{1}{(1-\mu)\kappa^d}M_S(\rho, 0)=K$" read "$\frac{1}{(1-\mu)\rho^d}M_S(\rho, 0)=K$".